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THE USE OF SINGULAR INTEGRALS IN WAVE PROPAGATION PROBLEMS; WITH APPLICATION TO THE POINT SOURCE IN A SEMI-INFINITE ELASTIC MEDIUM

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ABSTRACT

This is an investigation of the field due to a general point source of energy in an isotropic, elastic solid with a free surface. The paper is divided into three parts.

In Part I we are concerned with the development of new plane wave representations for the fundamental solutions of elastodynamics. There are two types of situation involved; we have the simpler type involved in the case of a steady point source which moves steadily with any constant velocity in an elastic medium, this type involves superposition of plane waves with respect to a single parameter, and we have the more complicated transient problem in which a point source is set up at a given moment, and thereafter moves at constant velocity, without change of strength.

In Part II we make use of the new representation for the field of a steadily moving source in the calculation of fields and displacements in the presence of a free surface and in Part III we do the same for the transient source. We discuss in some detail the application of the new approach to the case of a vertical load, to a horizontal load, and to a couple of arbitrary orientation, and we give a general discussion of the singularities to be expected for the general point source.

CONTENTS

THE USE OF SINGULAR INTEGRALS IN WAVE PROPAGATION PROBLEMS; WITH APPLICATION TO THE POINT SOURCE IN A SEMI-INFINITE ELASTIC MEDIUM.

M. Papadopoulos

General introduction:

In this paper we shall consider the formation and the propagation of tremors on the plane surface of an elastic solid. Although considerable research has been performed in this subject, (see a complete bibliography for pre-1955 work listed by Ewing, Jardetzky and Press, (1957), and for later work listed in the 1962 edition of Cagniard), the work to be described here is more general than that of Pekeris, or of Cagniard (1939), and it introduces a technique which has been found useful in the study of diffraction (Papadopoulos 1963e).

With others, Pekeris (1955a, 1955b, 1957, 1958) has solved a number of problems, each concerned with the setting-up of a point source of energy by the sudden application of a constant force or couple. Both Pekeris, and Cagniard (1939), have introduced the same kind of mathematical method. Cagniard, however, limited himself to situations with an axis of symmetry normal to the free surface, whereas Pekeris (1958) has given one example, that of the buried torque-pulse, where this property of symmetry is absent. In his paper Pekeris derived the form for the vertical component of the surface displacement without giving any of the detailed results for the horizontal components. Moreover, although he used

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transform methods to develop his solutions, his avoidance of the delta function in the definition of the point force and in the calculation of its associated field introduces a note of inconsistency in the mind of the reader.

The problem to be solved here involves, in its most general form, the sudden appearance of a point source which is moving with an arbitrary constant velocity. There is no restriction on this velocity, beyond that implied in not allowing this forced action to carry the source through the surface without specification of further source behaviour. The prototype of the energy source is a point force of step function time dependence and of arbitrary direction. Apart from implying that the most general source can be obtained from the point force by a series of linear operations (vector addition, differentiation and integration), we have no restriction on the type of source, and we have no restrictions on symmetry or orientation.

If the moving source is taken in an infinite medium without a free surface, the extention of known fixed source solutions to describe the moving field is simple. This is because the field is composed of sheer and compressional fields which travel independently and which may be described in terms of retarded potentials (de Hoop 1958). It is also easy to take a specific source and to find the moving field by transform methods as described by Eason, Fulton and Sneddon (1956). When coupling between shear and compression effects is enforced at a free surface, retarded potentials are certainly of no use, and the most direct approach seems to involve taking the known field of a transient source at a fixed point, and then superimposing the effects of a staggered distribution of such sources along the line of motion. Payton (1962) has used such a method, but when it is applied to the point source moving with no acceleration, it seems to be an overcomplicated approach. Physical interest in the case of a source which moves on a surface is quite clear; for source motion inside a solid, there is interest not only in possible earthquake motions, but in the effects of lines of explosive charge due to the finite propagation velocity of the explosion, and even in the enveloping effects of a sequence of discrete explosions as used in sequential rock blasting.

The method to be described here will, in its generality, involve us in the calculation of known results for the fixed source. In the process we find that the results of Pekeris and Lifson (1957), for the vertical force are incorrect in the vertical displacement component. The method is of special interest because specific parts of the field are picked out without ambiguity; not only can we pick out the singular parts of the field for the general point source, but we can pick out the stages in the development of the head wave field as well.

There are three stages in the discussion. In Part 1, it is shown that we may, in an infinite solid, represent the effect of a point force by an integral superposition of plane waves. Both real and complex plane waves have a part in these superpositions. The geometrical envelope of the waves linked with a specific source forms a singular surface, or wave front, and within this front the field at every point is found to be defined in terms of values on complex conjugate characteristics, while the field outside is similarly associated with real characteristics (or plane waves). In separate sections we take the case of the steady source which moves with constant velocity, this being a steady state problem in a moving system, and the case of the source which appears and then moves steadily. The first of these leads to integral superpositions of plane waves with respect to one parameter; it is described in some detail in order to simplify the description of the transient problem in which two integration parameters are needed.

For those who are familiar with the problems of wave propagation, the value of having a plane wave representation for a source field is that the effect of the plane surface may be found by treating each component wave as being reflected and refracted independently of the other component wave. A formal superposition of these reflected and refracted waves gives the total effect of the free surface. In Part 2 we discuss the structure of the field of the steadily moving source, when it travels horizontally, and we calculate general details of the surface displacement field. The special case of a fully supersonic point load moving on the surface has been described previously (Papadopoulos 1963a).

In Part 3 we examine the unsteady source field. We discuss in detail the surface displacement for a vertical force, for a horizontal force, and for a 'double force with moment' with a specific orientation, this being regarded by seismologists (Keilis-Borok et al., 1960) as a typical model for the focus of a tectonic earthquake. The complete displacement field for an arbitrary point source may not be found without quadrature, but explicit algebraic forms for all components are derived both for the initial displacements linked with the arrival of primary P and S waves, and for those linked with the arrival of head waves of shear, and for the singular surface waves due independently to the compressional and the shear part of the primary source when on the surface. Of special interest is the discussion of the relative strength of all the singularities which may appear.

Part 1: Plane wave representations for fundamental solutions in elastodynamics.

Section 1: The steadily moving source

The method to be described here was developed in the study of acoustic energy sources (Papadopoulos 1963b). A point source of unit strength moves along the z-axis with a constant velocity a; the velocity potential ϕ associated with such a singularity is the solution of the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right]\phi = -\delta(x)\,\delta(y)\,\delta(z-at) = -\delta(x)\,\delta(y)\,\delta(t-z/a)/a\ . \quad (1a)$$

Since this situation involves a steady motion in the z-direction we may infer that the velocity potential depends only on three independent variables, namely x, y and $\tau = t - z/a$. Equation (1a) now reduces to the form

$$\left[\frac{1}{\gamma^2}\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\phi = -\delta(x)\,\delta(y)\,\delta(\tau)/a\ , \quad (1b)$$

with $\gamma = ac(a^2-c^2)^{-\frac{1}{2}}$; in equation (1b) we can see the well-known change from hyperbolic to elliptic form as we permit a to pass from supersonic values, with $a > c$, to subsonic values with $a < c$. The equations (1) are appropriate for any real value of a. When $a = 0$ we have a static situation, and when $a = \infty$, we have, after multiplying the right hand side by a, the equation defining the potential of an infinite line source of unit strength per unit length.

For supersonic values of a, the transverse velocity of propagation, γ, is real, and the solution of equation (1b) has been given in the form

$$\phi = -\frac{1}{4\pi^2 a} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma}{(1-p^2)^{\frac{1}{2}}[\gamma\tau - px - (1-p^2)^{\frac{1}{2}} \, y \, \mathrm{sgn}\, y]} \, dp \; . \tag{2a}$$

For subsonic values of a, the transverse velocity is imaginary ($\gamma = i\mu$, say,) and the solution of equation (1b) has been given in a slightly different form with

$$\phi = \frac{1}{4\pi^2 a} \, \mathrm{Rl} \int_{-i\infty}^{i\infty} \frac{\mu}{(1-p^2)^{\frac{1}{2}} \, [i\mu\tau - xp - y \, \mathrm{sgn}\, y (1-p^2)^{\frac{1}{2}}]} \, dp \; . \tag{2b}$$

The main value of these representations is that away from the source, the wave nature of the associated field is evident. Corresponding expressions associated with strain nucleii will now be considered.

The eventual problem to be examined involves the setting up of elastic disturbances in a homogeneous isotropic medium, of density ρ and with Lamé constants λ and μ, in a half space $y \leq 0$ with the plane $y = 0$ a free surface. We set up the velocity vector $\underline{v}$ in terms of velocity potentials ϕ and ψ $(= A\underline{i} + B\underline{j} + C\underline{k})$, such that

$$\underline{v} = \nabla\phi + \nabla \times \psi \; , \tag{3a}$$

and

$$\nabla \cdot \psi = 0 \; . \tag{3b}$$

Given a body force $\underline{F}$, the stress-strain relations and the equations of motion reduce to the forms

$$\left[\frac{\partial^2}{\partial t^2} - c_1^2 \nabla^2\right] \nabla^2 \phi = \frac{\partial}{\partial t} \nabla \cdot \underline{F}$$

and

$$\left[\frac{\partial^2}{\partial t^2} - c_2^2 \nabla^2\right] \nabla^2 \underline{\psi} = \frac{\partial}{\partial t} \nabla \times \underline{F} \ , \tag{4}$$

with $\rho c_1^2 = \lambda + 2\mu$, $\rho c_2^2 = \mu$, c_1 and c_2 being the velocities of propagation of P and S waves respectively. When a force $\rho(X\underline{i} + Y\underline{j} + Z\underline{k})$ is applied steadily to the moving point $x = y = 0$, $z \leq at$, $\underline{F}$ has the form given by the equation

$$\underline{F} = (X\underline{i} + Y\underline{j} + Z\underline{k})\ \delta(x)\ \delta(y)\ \delta(z-at) \ ,$$

$$= (X\underline{i} + Y\underline{j} + Z\underline{k})\ \delta(x)\ \delta(y)\ \delta(\tau)/a \ , \tag{5}$$

and the equations (4) reduce to the transverse forms

$$c_1^2 \left[\frac{1}{\gamma_1^2} \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right] \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a^2} \frac{\partial^2}{\partial \tau^2}\right] \phi = \frac{\partial}{\partial \tau} \nabla \cdot \underline{F} \ , \tag{6a}$$

and

$$c_2^2 \left[\frac{1}{\gamma_2^2} \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial x} - \frac{\partial^2}{\partial y^2}\right] \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a^2} \frac{\partial^2}{\partial \tau^2}\right] \underline{\psi} = -\frac{\partial}{\partial \tau} \nabla \times \underline{F} , \tag{6b}$$

with $\gamma_1 = ac_1(a^2 - c_1^2)^{-\frac{1}{2}}$, and $\gamma_2 = ac_2(a^2 - c_2^2)^{-\frac{1}{2}}$.

Equation (6a) has a simple solution, given as the inverse of a triple Laplace transform by the equation

$$\phi = \frac{a^2\gamma_1^2}{ac_1^2(2\pi i)^3}\iiint_{-i\infty}^{i\infty} \frac{\exp[s\tau+\lambda x+\mu y][\lambda X+\mu Y-sZ/a]s\,ds\,d\lambda\,d\mu}{[s^2-\gamma_1^2(\lambda^2+\mu^2)][s^2+a^2(\lambda^2+\mu^2)]} \quad ; \tag{7}$$

but because the integrand is homogeneous in the three transform variables we may put $\lambda = -sp/\gamma_1$, $\mu = -sq/\gamma$, so as to reduce equation (7) to the form

$$\phi = -\frac{\gamma_1}{ac_1^2(2\pi i)^3}\int_{-i\infty}^{i\infty} ds \iint_{-i\infty}^{i\infty} \frac{(pX+qY+LZ)\exp s[\gamma_1\tau-px-qy]/\gamma_1}{(1-p^2-q^2)(p^2+q^2+L^2)}\,dp\,dq \quad ,$$

with $L = \gamma_1/a$. The integration with respect to s may now be carried out, so that

$$\phi = \frac{-\gamma_1}{c_1^2\,a(2\pi i)^2}\iint_{-\infty}^{\infty}\frac{(pX+qY+LZ)\,\delta[\gamma_1\tau-px-qy]/\gamma_1}{(1-p^2-q^2)(p^2+q^2+L^2)}\,dp\,dq \quad ; \tag{8}$$

the presence of the delta function is meaningful only as long as the function $(\gamma_1\tau-px-qy)/\gamma_1$ is real. In the work being considered it is not always possible to assume this, and the delta function must be regarded as having an alternative singular form, with (8) taking the form

$$\phi = \frac{-\gamma_1^2}{4\pi^2 c_1^2 a}\,\mathrm{Rl}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i(pX+qY+LZ)}{\pi(1-p^2-q^2)(p^2+q^2+L^2)(\gamma_1\tau-px-qy)}\,dp\,dq \;. \tag{9}$$

Away from the point singularity, ϕ satisfies the wave equation, and the integration parameters must be restricted to the circle $p^2+q^2=1$. Thus for arbitrary p, q may take only the values $q = \pm(1-p^2)^{\frac{1}{2}}$, and in turn this means that the q-integration involves only the calculation of residues at these points. We specify the branch of $(1-p^2)^{\frac{1}{2}}$ by taking the p-integration path along the real p-axis, with indentations taking it above the point $p = 1$ and below the point $p = -1$. With this convention we have

$$\phi = \frac{1}{4\pi^2 a} \text{ Rl} \int_{-\infty}^{\infty} \frac{[pX + (1-p^2)^{\frac{1}{2}} \operatorname{sgn} y\, Y + LZ}{(1-p^2)^{\frac{1}{2}} [\gamma_1\tau - px - y \operatorname{sgn} y (1-p^2)^{\frac{1}{2}}]} \, dp \ . \tag{10a}$$

or, following a change in scale of the integration variable

$$\phi = \frac{1}{4\pi^2 a} \text{ Rl} \int_{-\infty}^{\infty} \frac{pX + (M^2-p^2)^{\frac{1}{2}} Y \operatorname{sgn} y + LMZ}{(M^2-p^2)^{\frac{1}{2}} [\gamma_2\tau - px - (M^2-p^2)^{\frac{1}{2}} y \operatorname{sgn} y]} \, dp \tag{11a}$$

with $M = \gamma_2/\gamma_1 < 1$. These representations are only correct for fully supersonic source motion, with $a > c_1$. For the range $a < c_1$, the transverse velocity γ_1 is imaginary, with $\gamma_1 = i\mu_1$, say, and the results of simplifying the integral (7) are slightly different, taking the integration paths for both p and q along the imaginary axis instead of the real axis. To correspond to the formula (10a) we have the integral

$$\phi = -\frac{1}{4\pi^2 a} \text{ Rl} \int_{-i\infty}^{i\infty} \frac{i[pX + (1-p^2)^{\frac{1}{2}} Y \operatorname{sgn} y + LZ]}{(1-p^2)^{\frac{1}{2}} [i\mu_1\tau - px - (1-p^2)^{\frac{1}{2}} y \operatorname{sgn} y]} \, dp \ . \tag{10b}$$

The integral (11a) keeps its form when $a > c_2$, but it changes to

$$\phi = -\frac{1}{4\pi^2 a}\,\mathrm{Ri}\int_{-i\infty}^{i\infty}\frac{i\,[pX + (M^2-p^2)^{\frac{1}{2}}\,Y\,\mathrm{sgn}\,y + LMZ]}{(M^2-p^2)^{\frac{1}{2}}\,[i\mu_2\tau - px - (M^2-p^2)^{\frac{1}{2}}\,y\,\mathrm{sgn}\,y]}\,dp \qquad (11b)$$

in the case of fully subsonic source motion, with $a < c_2$; when $\gamma_2 = i\mu_2$.

The three cartesian components of $\underline{\psi}$ satisfy equation (6b) and may be written in the form

$$\underline{\psi} = \frac{1}{4\pi^2 a}\,\mathrm{Ri}\int_{-\infty}^{\infty}\frac{[LMY - Z(1-p^2)^{\frac{1}{2}}\,\mathrm{sgn}\,y]\,\underline{i} + [pZ - LMX]\,\underline{j} + [X(1-p^2)^{\frac{1}{2}}\,\mathrm{sgn}\,y - pY]\,\underline{k}}{(1-p^2)^{\frac{1}{2}}\,[\gamma_2\tau - px - (1-p^2)^{\frac{1}{2}}\,y\,\mathrm{sgn}\,y]}\,dp \qquad (12a)$$

when $a > 0_2$, or

$$\underline{\psi} = \frac{1}{4\pi^2 a}\,\mathrm{Ri}\int_{-i\infty}^{i\infty}\frac{i\{[LMY - Z(1-p^2)^{\frac{1}{2}}\,\mathrm{sgn}\,y]\,\underline{i} + [pZ - LMX]\,\underline{j} + [X(1-p^2)^{\frac{1}{2}}\,\mathrm{sgn}\,y - pY]\,\underline{k}\}}{(1-p^2)^{\frac{1}{2}}\,[i\mu_2\tau - px - (1-p^2)^{\frac{1}{2}}\,y\,\mathrm{sgn}\,y]}\,dp \qquad (12b)$$

when $a < c_2$.

Notice here that as the source velocity is varied and the transverse velocities take positive imaginary values, the other velocity parameters, L and M, may also take imaginary values.

The potentials which correspond to other steadily moving strain nucleii may easily be constructed by superimposing the results for simple forces. Thus two equal forces acting to oppose each other in the x-direction but with a moment about the z-axis may be taken in a limiting form as a 'double force with moment'; the potentials in this case are

$$\phi = -\frac{\operatorname{sgn} y}{4\pi^2 a\gamma_2}\frac{\partial}{\partial t}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{p}{[\gamma_2\tau - px - (M^2-p^2)^{\frac{1}{2}}\, y \operatorname{sgn} y]}\,dp$$

and

$$\underline{\psi} = -\frac{\operatorname{sgn} y}{4\pi^2 a\gamma_2}\frac{\partial}{\partial t}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{[(1-p^2)^{\frac{1}{2}}\operatorname{sgn} y\,\underline{k} - LM\,\underline{i}\,]}{[\gamma_2\tau - px - (1-p^2)^{\frac{1}{2}}\, y \operatorname{sgn} y]}\,dp\ .$$

Similarly a combination of three equal orthogonal double forces without moment gives the potentials

$$\phi = \frac{1}{4\pi^2 a\gamma_2}\frac{\partial}{\partial t}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{M^2(1+L^2)}{(M^2-p^2)^{\frac{1}{2}}[\gamma_2\tau - px - (M^2-p^2)^{\frac{1}{2}}\, y \operatorname{sgn} y]}\,dp$$

and $\underline{\psi} = 0$. This being a constant real multiple of the time derivative of the potential for an acoustic source, we might concentrate attention on this acoustic source as being a simple case which does not produce shear, while to take account of shear the simplest case is that of the simple force. A detailed examination of other strain nucleii (e.g., as listed by Love (1927) in the study of elastostatics, or by Keilis-Borok (1960) in the study of seismology) is not needed here. Thus in addition to the potentials (11) and (12) we have to refer to the potentials of a moving source of unit strength, in the form

$$\phi = -\frac{1}{4\pi^2 a}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{\gamma^2}{(M^2-p^2)^{\frac{1}{2}}\,[\gamma_2\tau - p\lambda - (M^2-p^2)^{\frac{1}{2}}\, y \operatorname{sgn} y]}\,dp \qquad (13a)$$

when $a > c_2$, or

$$\phi = \frac{1}{4\pi^2 a} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\mu_2}{(M^2 - p^2)^{\frac{1}{2}} [i\mu_2 \tau - px - (M^2 \; p^2)^{\frac{1}{2}} \; y \, \mathrm{sgn} \, y]} \, dp \; . \tag{13b}$$

Each of the integrals given may be reduced to an explicit formula. The process involves shifting the formal integration path to the curve on which the wave function of the denominator is real. On this path the integral is singular, the Cauchy principal value is either zero or imaginary, and the only real contribution to the potentials comes from the residue at points for which the wave function vanishes. For supersonic motion there are two conjugate zeros, while for the subsonic case there is only one. Real zeros of the wave function do not, in the integrals given, provide real residue contributions to the potentials. Each one of the integrals has a singular geometry. In all cases, the source point is a singular point of the integral, and in the supersonic cases there is a singular (conical) surface to contain the source field. These singularities are determined by the simultaneous vanishing of the denominators of the integrals and their derivatives with respect to p. The conical surface is seen to be the envelope of plane waves generated by the source. The property that complex zeros of the wave function are the only ones to provide residue terms that matter for the source fields is equivalent to saying that these fields are determined by complex plane characteristic surfaces passing through each point of observation. The details are similar to those given previously (Papadopoulos 1963b).

Each plane wave in one of the given singular integrals is reflected and refracted at a plane boundary just as if it were a real plane wave. In order to fix ideas we shall consider the case of a point source moving at a uniform depth h below the free surface of an elastic solid in the plane $y = 0$. Away from the source point the potentials must satisfy the equations

$$\left[\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right]\phi = 0 \ ,$$
$$\left[\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right]\underline{\psi} = 0 \ , \tag{14a}$$

and

$$\nabla \underline{\psi} = A_x + B_y + C_z = 0 \ , \tag{14b}$$

for $y \leq 0$, while the conditions of vanishing stress at the surface are that

$$\frac{\partial}{\partial t} T_{yy} = \lambda \nabla^2 \phi + 2\mu [\phi_{yy} + A_{zy} - C_{xy}] = 0 \ , \tag{15a}$$

$$\frac{\partial}{\partial t} T_{yx} = \mu [2\phi_{yx} + A_{xz} - C_{xx} + C_{yy} - B_{zy}] = 0 \ , \tag{15b}$$

$$\frac{\partial}{\partial t} T_{yz} = \mu [2\phi_{yz} + A_{zz} - A_{yy} - C_{zx} + B_{xy}] = 0 \ ,$$

for $y = 0$. The third condition may be combined with (15b),(14b) and the wave equation to give the simpler form

$$\frac{\partial^2}{\partial t^2} B_y = 0 \tag{15c}$$

which implies that the potential B is an even function of y about the plane $y = 0$.

In the absence of boundaries the source field is of two slightly different forms. For $y > -h$, we have a general source field in the form

$$\phi = \mathrm{Rl}\int \frac{F}{(M^2-p^2)^{\frac{1}{2}}[\gamma_2\tau - px - (M^2-p^2)^{\frac{1}{2}}(y+h)]}\,dp,\quad \underline{\psi} = \mathrm{Rl}\int \frac{A\underline{i} + B\underline{j} + C\underline{k}}{(1-p^2)^{\frac{1}{2}}[\gamma_2\tau - px - (1-p^2)^{\frac{1}{2}}(y+h)]}\,dp \tag{16a}$$

with

$$pA + LMC + (1-p^2)^{\frac{1}{2}} B = 0 \ .$$

For $y < -h$, there is the corresponding form

$$\phi = \mathrm{Rl}\int \frac{F^*}{(M^2-p^2)^{\frac{1}{2}}[\gamma_2\tau - px + (M^2-p^2)^{\frac{1}{2}}(y+h)]}\,dp,\quad \underline{\psi} = \mathrm{Rl}\int \frac{A^*\underline{i} + B^*\underline{j} + C^*\underline{k}}{(1-p^2)^{\frac{1}{2}}[\gamma_2\tau - px + (1-p^2)^{\frac{1}{2}}(y+h)]}\,dp \tag{16b}$$

with

$$pA^* + LMC^* - (1-p^2)^{\frac{1}{2}} B^* = 0 \ .$$

It is presumed that the functions F, A, B, C, F^*, A^*, B^* and C^* are real functions of p when the transverse velocity parameters are real. The integrals (11), (12) and (13) are those we have specially in mind to provide explicit forms for the numerators of the integrals (16), but there is no restriction to these. We may also consider derivatives of these integrals with respect to t, just as we may consider more general linear operations with respect to t as may be needed to provide results for sources which vary with time. From the conditions at $y = 0$, it follows that we may write down the scattered field in $y < 0$ in the form

$$\phi = \mathrm{Rl} \int \frac{1}{(M^2-p^2)^{\frac{1}{2}} R(p)} \left\{ \frac{F_p}{\gamma_2 \tau - px + (M^2-p^2)^{\frac{1}{2}}(y-h)} + \frac{F_s}{\gamma_2 \tau - px + (M^2-p^2)^{\frac{1}{2}} y - (1-p^2)^{\frac{1}{2}} h} \right\} dp ,$$

$$\underline{\psi} = \mathrm{Rl} \int \frac{1}{(1-p^2)^{\frac{1}{2}} R(p)} \left\{ \frac{A_p \underline{i} + B_p \underline{j} + C_p \underline{k}}{\gamma_2 \tau - px + (1-p^2)^{\frac{1}{2}} y - h(M^2-p^2)^{\frac{1}{2}}} + \frac{A_S \underline{i} + B_S \underline{j} + C_S \underline{k}}{\gamma_2 \tau - px + (1-p^2)^{\frac{1}{2}}(y-h)} \right\} dp \tag{17}$$

where

$$R(p) = (1 - L^2 M^2 - 2p^2)^2 + 4(1-p^2)^{\frac{1}{2}}(p^2 + L^2 M^2) , \tag{18a}$$

the scattering coefficients linked with the primary P field are defined by the equations

$$\left. \begin{aligned} F_p + F\, R(p) &= 8(1-p^2)^{\frac{1}{2}}(p^2 + L^2 M^2)\, F , \\ C_p = -pA_p/LM &= -4p(1-p^2)^{\frac{1}{2}}(1 - L^2 M^2 - 2p^2)\, F \end{aligned} \right\} \tag{18b}$$

and

$$B_p = 0 ,$$

while the scattering coefficients linked independently with the primary S field are defined by the equations

$$\left.\begin{aligned} F_S &= 4(M^2-p^2)^{\frac{1}{2}}(1-L^2M^2-2p^2)\,(pC-LMA)\ , \\ B_S &= B\,R(p) = \quad -R(p)\,(LMC+pA)\,(1-p^2)^{-\frac{1}{2}}\ , \\ [C_S+C\,R(p)] &= -p[A_S+A\,R(p)]/LM = 8p(M^2-P^2)^{\frac{1}{2}}(1-p^2)^{\frac{1}{2}}(pC-LMA) \end{aligned}\right\} . \tag{18c}$$

The integrals (17) represent the extra field which has to be added to the primary field (16a) in the region $0 > y > -h$, and to the primary field (16b) in the region $y < -h$. The field expressions for a source of the same form travelling on the free surface are to be found by taking the results in the region $y < -h$ in the limit as $h \to 0$.

A general discussion of these results will be given in Part 2.

Section 2: The transient point source.

We have dealt in Section 1 with the fields of steadily moving sources; these fields have been derived in the form of integrals of singular plane waves with respect to a single parameter. To give similar forms for transient point source fields involves us in integrals of singular plane waves with respect to two parameters.

We shall consider first the solution of the equations (4) when the body force $\underline{F}$ is given by

$$\underline{F} = (X\underline{i} + Y\underline{j} + Z\underline{k})\,\delta(x)\,\delta(y)\,\delta(z)\,U(t)\ . \tag{19}$$

The equation for the scalar potential ϕ has the solution

$$\phi = \frac{1}{(2\pi i)^4} \iiiint_{-i\infty}^{i\infty} \frac{\exp[st + \lambda x + \mu y + \nu z](\lambda X + \mu Y + \nu Z)}{[s^2 - c_1^2(\lambda^2 + \mu^2 + \nu^2)][\lambda^2 + \mu^2 + \nu^2]}\, ds\, d\lambda\, d\mu\, dw\ , \tag{20}$$

this being the inversion of a quadruple Laplace transform. The integrand is a homogeneous function of the four transform variables, and we may therefore put $\lambda = -s\alpha_1/c_1$, $\mu = -s\alpha_2/c_1$ and $\gamma = -s\alpha_3/c_1$ to derive the expression

$$\phi = \frac{1}{(2\pi i)^4} c_1^2 \iiint_{-\infty}^{\infty} \frac{(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)}{(1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)}\, d\alpha_1\, d\alpha_2\, d\alpha_3 \int_{-i\infty}^{i\infty} \exp s[c_1 t - \alpha_1 x - \alpha_2 y - \alpha_3 z]/c_1$$

$$= -\frac{1}{(2\pi)^3 c_1^2} \mathrm{Ri} \iiint_{-\infty}^{\infty} \frac{(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)\, c_1}{\pi(1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(c_1 t - \alpha_1 x - \alpha_2 y - \alpha_3 z)}\, d\alpha_1\, d\alpha_2\, d\alpha_3\ .$$

This expression satisfies the wave equation away from the origin, and for given α_1, α_3, α_2 is restricted to one of the values $\pm\beta = \pm(1-\alpha_1^2-\alpha_3^2)^{\frac{1}{2}}$. Thus the whole integration with respect to α_2 must involve only residues at these two points. Here we define the branch of the radical so that the real α_2-axis passes, with indentations as required, above the point $\alpha_2 = \beta$, and below the point $\alpha_2 = -\beta$. These poles give the residue contribution

$$\phi = -\frac{1}{8\pi^3 c_1} \mathrm{Ri} \iint_{-\infty}^{\infty} \frac{i(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)\ \mathrm{sgn}\, y}{\alpha_2 [c_1 t - \alpha_1 x - \alpha_3 z - \alpha_2 y]}\, d\alpha_1\, d\alpha_3 \ , \tag{21a}$$

with $\alpha_2 = \beta\ \mathrm{sgn}\ y$, or, on dividing both integration variables by $m = c_2/c_1$,

$$\phi = -\frac{1}{8\pi^3 c_2} \mathrm{Ri} \iint_{-\infty}^{\infty} \frac{i(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)\ \mathrm{sgn}\ y}{\alpha_2 (c_2 t - \alpha_1 x - \alpha_2 y - \alpha_3 z)}\, d\alpha_1\, d\alpha_3 \tag{21b}$$

with $\alpha_2 = \alpha\,\mathrm{sgn}\,y$, $\alpha = [m^2 - \alpha_1^2 - \alpha_3^2]^{\frac{1}{2}}$.

In this integral it is required that the 'slowness' variables be represented by a point on a spherical slowness surface of radius m . The integral is therefore invariant under rotation of these variables, this rotation may be chosen to simplify further calculations.

The equation (4), which defines the vector potential $\underline{\psi}$, may be solved in a similar fashion to give

$$\underline{\psi} = -\frac{1}{8\pi^3 c_2} \mathrm{Ri} \iint_{-\infty}^{\infty} \frac{i\,\mathrm{sgn}\,y\,[(Y\alpha_3 - Z\beta_2)\underline{i} + (Z\alpha_1 - X\alpha_3)\underline{j} + (X\beta_2 - Y\alpha_1)\underline{k}]}{(c_2 t - \alpha_1 x - \beta_2 y - \alpha_3 z)\,\beta_2}\, d\alpha_1\, d\alpha_3 \tag{22}$$

with $\beta_2 = \beta \operatorname{sgn} y$. Note that the wave functions of both the integrals (21b) and (22) are identical in their behaviour on the plane $y = 0$. This property has been chosen deliberately to simplify the matching of P and S waves at the free surface.

An expression of similar form for the scalar potential of a dilatation source of unit strength is

$$\phi = \frac{1}{8\pi^3 c_2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{1}{\alpha(c_2 t - \alpha_1 x - \alpha_3 z - \alpha y \operatorname{sgn} y)} \, d\alpha_1 \, d\alpha_3 \quad . \tag{23}$$

Simple extensions of these formulae may be obtained to cover the case of a transient source which moves with constant velocity. In this situation the body force is

$$\underline{F} = (X\underline{i} + Y\underline{j} + Z\underline{k})\,\delta(x - v_1 t)\,\delta(y - v_2 t)\,\delta(z - v_3 t)\,U(t) \quad . \tag{24}$$

The effect of the motion is to introduce into the integrand (20) an extra homogeneous factor $s(s + \lambda v_1 + \mu v_2 + \nu v_3)^{-1}$. The details of the subsequent evaluation are not changed; the motion of the source leads us to include in the integrals (21b) and (23) and extra factor

$$V_1 = c_2(c_2 - \alpha_1 v_1 - \alpha_2 v_2 - \alpha_3 v_3)^{-1} \tag{25a}$$

while the integral 22 has the extra factor

$$V_2 = c_2(c_2 - \alpha_1 v_1 - \beta_2 v_2 - \alpha_3 v_3)^{-1} \tag{25b}$$

With integration paths already chosen as curves on which the wave function is real the general set of potentials associated with the appearance of a source at the initial point $x = z = 0$, $y = -h$ may be written in the form

$$\phi = \mathrm{Rl} \iint \frac{i F V_1}{\alpha[c_2 t - \alpha_1 x - \alpha_3 z - \alpha(y+h)]} \, d\alpha_1 \, d\alpha_3 \ ,$$

$$\underline{\psi} = \mathrm{Rl} \iint \frac{i[A\underline{i} + B\underline{j} + C\underline{k}]\, V_2}{\beta[c_2 t - \alpha_1 x - \alpha_3 z - \beta(y+h)]} \, d\alpha_1 \, d\alpha_3 \qquad (26a)$$

for $y + h > 0$, and

$$\varphi = \mathrm{Rl} \iint \frac{i F^* V_1^*}{\alpha[c_2 t - \alpha_1 x - \alpha_3 z + \alpha(y+h)]} \, d\alpha_1 \, d\alpha_3$$

$$\underline{\psi} = \mathrm{Rl} \iint \frac{i[A^*\underline{i} + B^*\underline{j} + C^*\underline{k}]V_2^*}{\beta[c_2 t - \alpha_1 x - \alpha_1 x - \alpha_3 z + \beta(y+h)]} \, d\alpha_1 \, d\alpha_3 \qquad (26b)$$

for $y + h < 0$, the two representations being needed to eliminate in the numerators the function $\mathrm{sgn}(y+h)$. (These expressions, of course, represent only potentials of degree -1, but to give a general form for a potential of degree $-n$ we need only differentiate the integrals (26) $(n-1)$-times with respect to t).

Given a primary source with potentials of this form, the effect of a free surface in the plane $y = 0$ is to introduce scattered potentials, in $y < 0$ is to introduce scattered potentials, in $y < 0$, given by

$$\phi = \mathrm{Rl}\iint\left\{\frac{F_p}{c_2t-\alpha_1x-\alpha_3z+\alpha(y-h)}+\frac{F_s}{c_2t-\alpha_1x-\alpha_2z+\alpha y-\beta h}\right\}d\alpha_1\,d\alpha_3 \quad (27a)$$

and

$$\underline{\psi} = \mathrm{Rl}\iint\frac{i}{R\beta}\left\{\frac{A_p\underline{i}+B_p\underline{j}+C_p\underline{k}}{c_2t-\alpha_1x-\alpha_3z+\beta y-\alpha h}+\frac{A_s\underline{i}+B_s\underline{j}+C_s\underline{k}}{c_2t-\alpha_1x-\alpha_2z+\beta(y-h)}\right\}d\alpha_1\,d\alpha_3. \quad (27b)$$

Then, with the function

$$R = [1-2\alpha_1^2-2\alpha_3^2]^2+4\,\alpha\beta(\alpha_1^2+\alpha_3^2)\quad , \qquad (28a)$$

appearing as a determinant of the boundary conditions (15), we find the scattered amplitudes of each individual plane wave from the equations

$$F_P + FRV_1 = 8\alpha\beta(\alpha_1^2+\alpha_3^2)\,F\,V_1$$

$$C_P = -A_P\alpha_1/\alpha_3 = -4\,\beta\alpha_1(1-2\alpha_1^2-2\alpha_3^2)\,F\,V_1$$

$$B_P = 0 \qquad (28b)$$

and

$$C_S + CRV_2 = -\alpha_1[A_S+ARV_2]/\alpha_3 = 8\,\alpha_1\alpha\beta[C\alpha_1-A\alpha_3]V_2\quad ,$$

$$F_S = 4(1-2\alpha_1^2-2\alpha_3^2)\,\alpha[C\alpha_1-A\alpha_3]V_2\quad ,$$

$$B_S = BRV_2 = -\beta R[A\alpha_1+C\alpha_3]V_2\quad . \qquad (28c)$$

The expressions (28b) and (28c) are quite independent, the former represent reflection and refraction coefficients linked with an incident P wave, and the latter are linked similarly with an incident S wave.

In the region $0 > y > -h$, the total field is given as the sum of the expressions (26a) and (27); in the region $y < -h$ it is likewise given as the sum of the expressions (26b) and (27). The case when the source is formed at the surface is obtained by taking the field in $y < -h$ and then allowing h to vanish.

One restriction remains to be mentioned here. Positive values of v_2 will bring the source to the surface when $0 < t < h/v_2$. The associated field expressions are therefore valid only in a restricted region, and a complete description of the subsequent field has to follow a choice of the subsequent source motion.

Part 2: The steadily moving source in an elastic solid with a free plane surface.

Section 1: The simplification of integral solutions, and the field structure.

We shall first examine the means for simplifying integrals given in general form as in equations (1.16) and (1.17), for the case of fully supersonic source motion. For the primary source potentials of equation (1.16), the evaluation is performed by shifting the integration path from the real p-axis to the curve on which the wave function of the denominator is real. Thus, for the primary dilatation field of equation 1.16a, the wave function

$$\gamma_2\tau - px - (y+h)(M^2 - p^2)^{\frac{1}{2}} \qquad (1)$$

is real on the hyperbola $p = M\cosh(w + i\theta)$ for $-\infty < w < \infty$, with $\theta = \arctan(y+h)/x$. The branch of $(M^2 - p^2)^{\frac{1}{2}}$ chosen has a positive real part, with $(M^2 - p^2)^{\frac{1}{2}} = -iM\sinh(w + i\theta)$, in order that the wave function (1) be part of a disturbance which moves in the direction of increasing y.

With θ chosen and the branch of the radical defined in this manner, it is clear that the wave function (1) does not vanish when $\tau < 0$. With given positive τ, it has complex conjugate zeros on the hyperbola $p = M\cosh(w+i\theta)$ for $0 < [x^2 + (y+h)^2]^{\frac{1}{2}} < \gamma_1\tau$, and it has real zeros if $[x^2 + (y+h)^2]^{\frac{1}{2}} > \gamma_1\tau$. It follows, having shifted the integration path from the real p-axis to the locus of complex zeros, that there are three distinct contributions to consider. First the shift in path involves us with residue contributions from real poles; for the primary source terms these residues are strictly imaginary, and do not contribute to the field. Second, the line integral along the hyperbola is singular (the path is shown in figure 1a), but the Cauchy principal value has a zero real part. This is because the integration path is symmetrical about the real p-axis, and the contribution to the real part from the lower half of the path annuls that from the upper part. The third contribution is the residue contribution from the complex poles; for the primary dilatation field these are at the points

$$p = \frac{\gamma_2 \tau x \pm i(y+h)\{\gamma_2^2\tau^2 - M^2[x^2 + (y+h)^2]\}^{\frac{1}{2}}}{x^2 + (y+h)^2} ,$$

and these appear only when $\gamma_1\tau > [x^2 + (y+h)^2]^{\frac{1}{2}} > 0$. The bounding surface for this residue contribution, a section of the cone $\gamma_1\tau = [x^2 + (y+h)^2]^{\frac{1}{2}} > 0$, is a singular surface of the dilatation field. It is associated with double zeros of the wave function, that is, with points for which both the wave function and its derivative with respect to p vanish: this is the precise definition of the envelope of the individual plane waves. It marks the boundary surface for the contribution of residues at complex poles, i.e., for the effects of the conjugate complex characteristics of the wave equation.

The primary potentials (1.16a) and (1.16b) reduce in this manner to the form

$$\left\{\gamma_2^2\tau^2 - M^2[x^2 + (y+h)^2]\right\}^{\frac{1}{2}} \phi/2\pi = \begin{cases} \mathrm{Rl}\,[F(p_1)] & \text{for } y+h > 0 \\ \mathrm{Rl}\,[F^*(p_1)] & \text{for } y+h < 0 , \end{cases} \qquad (2a)$$

with $\quad p_1[x^2+(y+h)^2]^{\frac{1}{2}} = \gamma_2\tau x - i(y+h)\{\gamma_2^2\tau^2 - M^2[x^2+(y+h)^2]\}^{\frac{1}{2}}\,\mathrm{sgn}(y+h)$,

and $\quad \gamma_1\tau = \gamma_2\tau/M > [x^2+(y+h)^2]^{\frac{1}{2}} > 0$,

while

$$\left[\gamma_2^2\tau^2 - x^2 - (y+h)^2\right]^{\frac{1}{2}}\ \underline{\psi}/2\pi = \begin{cases} \mathrm{Rl}\,[A(p)\,\underline{i} + B(p)\,\underline{j} + C(p)\,\underline{k}]_{p=p_2} & \text{for } y+h>0 \\ \mathrm{Rl}\,[A^*(p)\,\underline{i} + B^*(p)\,\underline{j} + C^*(p)\,\underline{k}]_{p=p_2} & \text{for } y+h<0 \end{cases} ,\tag{2b}$$

with

$$p_2\,[x^2+(y+h)^2]^{\frac{1}{2}} = \gamma_2\tau x - i(y+h)[\gamma_2^2\tau^2 - x^2 - (y+h)^2]^{\frac{1}{2}}\,\mathrm{sgn}(y+h)$$

and

$$\gamma_2\tau > [x^2+(y+h)^2]^{\frac{1}{2}} > 0 \ .$$

For completeness, it should also be mentioned here that the shift of integration path should involve contributions from the circle at infinity. These make no contribution to the field unless specific points in the vicinity of the source are being examined or if $\tau \to 0$.

The integrals (1.17) are a little more difficult to evaluate. They are distinguished by the presence of the function

$$R(p) = (1 - L^2M^2 - 2p^2)^2 + 4(1-p^2)^{\frac{1}{2}}(M^2-p^2)^{\frac{1}{2}}(p^2 + L^2M^2) \ . \tag{3}$$

This is a characteristic function defining the transverse velocity γ_R of Rayleigh waves. It has zeros at the points $p = \pm R = \gamma_2/\gamma_R$, where $\gamma_R = ac_R(a^2 - c_R^2)^{-\frac{1}{2}}$, this result being easily deduced from the limiting two dimensional situation when, with $a \to \infty$ and $m = c_2/c_1$,

$$R(p) \to (1-2p^2)^2 + 4p^2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}} = 0$$

is the equation with roots $p = \pm c_2/c_R$ defining the actual velocity v_R of Rayleigh waves.

The zeros of $R(p)$ are poles of the integrals (1.17) . They may only be linked with singular surfaces of the scattered field if they are actually poles of second, or higher, order, and this is the case only if the wave function of the denominator vanishes with $R(p)$. It is clear from the form of equation (3) that $R(p)$ will not vanish except for real values of p with $|p| > 1$. Other (complex) zeros which are found after $R(p)$ is made rational are not zeros which can be associated with the branches defined for the radicals. The wave functions for (1.17) may not vanish, however, except in the single case with $h = y = 0$, y being restricted to non-positive values in these expressions. Singular Rayleigh waves do not occur except, on the free surface, for sources set up on the surface.

Notice that the integrals (1.17) are of two types. The first integral for ϕ and the second integral for ψ are simple in structure, being integrals of plane waves originating at the singular image point $y = h$, $x = 0$, $z = at$. These integrals are, in turn, reflected P and S fields. They are evaluated in the same manner as the primary field expressions, by shifting the path of integration to the curve on which the wave function is real. For the P field there is nothing new to discuss; we have only residue contributions from conjugate complex poles to evaluate and this defines a field contained between the surface $y = 0$ and the image cone $\gamma_1\tau = [x^2 + (y-h)^2]^{\frac{1}{2}}$ for $\tau > 0$. The reflected S field contained within the cone $\gamma_2\tau = [x^2 + (y-h)^2]^{\frac{1}{2}}$ is obtained from the complex zeros of the wave function, but there is an extra complication to be examined. The function $R(p)$ contains branch points at $p = \pm M$, with $M < 1$; the locus of complex zeros of the wave function

$$\gamma_2\tau - px + (y-h)(1-p^2)^{\frac{1}{2}} \tag{4}$$

is bound to cross the real p-axis at points with $|p| < 1$ and $|p| > M$ if

$$\arccos M > |\arctan(y-h)/x| > 0 \quad .$$

Then, as shown in figure (1b), the shift of integration path will involve a horizontal singular loop integral about one or other of the points $p = \pm M$ as well as the singular integral along the hyperbola. Here we find that only residue terms from real zeros of the wave function (4) make a real contribution to the field, with the principal value of the line integral vanishing in its real part. These real zeros of the wave function provide real plane wave contributions to the reflected shear field, they are naturally outside the singular envelope of plane waves which contains the effects of the complex poles, and these head wave contributions appear only between the cone $\gamma_2\tau = [x^2 + (y-h)^2]^{\frac{1}{2}} > 0$, the plane $y = 0$ and the tangent planes $\gamma_2\tau = \pm Mx - (y-h)(1-M^2)^{\frac{1}{2}}$.

Thus the reflected P field in $y < 0$ is

$$\phi = \frac{2\pi}{\{\gamma_2^2\tau^2 - M^2[X^2+(y-h)^2]\}^{\frac{1}{2}}} \ \mathrm{Rl}\left\{\frac{F_P(p_1)}{R(p_1)}\right\}, \tag{5a}$$

for

$$p_1[x^2+(y-h)^2]^{\frac{1}{2}} = \gamma_2\tau x + i(y-h)\left\{\gamma_2^2\tau^2 - M^2[x^2+(y-h)^2]\right\}^{\frac{1}{2}} ,$$

when

$$\gamma_1\tau > [x^2+(y-h)^2]^{\frac{1}{2}} > 0 .$$

The part of the reflected shear field inside the reflected shear cone is

$$\underline{\psi} = \frac{2\pi}{[\gamma_2^2\tau^2 - x^2 - (y-h)^2]^{\frac{1}{2}}} \ \mathrm{Rl}\left\{\frac{A_S(P_2)\underline{i} + B_S(p_2)\underline{j} + C_S(p_2)\underline{k}}{R(p_2)}\right\} , \tag{5b}$$

for

$$p_2[x^2+(y-h)^2]^{\frac{1}{2}} = \gamma_2\tau x + i(y-h)[\gamma_2^2\tau^2 - x^2 - (y-h)^2]^{\frac{1}{2}} ,$$

when

$$\gamma_2\tau > [x^2+(y-h)^2]^{\frac{1}{2}} > 0 .$$

The associated head wave contribution is

$$\underline{\psi} = \frac{2\pi}{[x^2+(y+h)^2 - \gamma_2^2\tau^2]^{\frac{1}{2}}} \ \mathrm{Im}\left[\frac{A_S(p)\underline{i} + B_S(p)\underline{j} + C_S(p)\underline{k}}{R(p)}\right]_{\substack{p=p_H \\ (m^2-p_H^2)^{\frac{1}{2}} = i(p_H^2-m^2)^{\frac{1}{2}}}} \tag{5c}$$

where p_H is that value of the roots

$$\frac{\gamma_2 \tau x \pm (y-h)[x^2+(y-h)^2-\gamma_2^2\tau^2]^{\frac{1}{2}}}{x^2+(y-h)^2}$$

which lies between the point $p = x[x^2+(y-h)^2]^{-\frac{1}{2}}$ and the origin.

Now notice the following property. In any given section of the transverse plane, the reflected dilatation cone does not appear below the free surface, with its interior field contributions, until $\tau = h/\gamma_1$. Similarly, the reflected shear cone appears below the surface when $\tau > h/\gamma_2$. On the other hand, the planes $\gamma_2\tau = \pm Mx-(y-h)(1-M^2)^{\frac{1}{2}}$, which are tangent to this reflected shear cone and which are the fronts of the head wave contribution can not appear below the free surface until $\gamma_2\tau = h(1-M^2)^{-\frac{1}{2}}$, when $p = \pm M$ is a double zero of (4). There is therefore a period when the reflected shear field is present without a head wave contribution. The two stages in the growth of this field are shown in figure (2).

The remaining terms in the integral (1.17) are more difficult to evaluate. This is because the refracted wave surfaces which are envelopes of the plane P waves of form $\gamma_2\tau - px + y(M^2-p^2)^{\frac{1}{2}} - h(1-p^2)^{\frac{1}{2}}$ and of the plane S waves of form $\gamma_2\tau - px + y(1-p^2)^{\frac{1}{2}} - h(M^2-p^2)^{\frac{1}{2}}$ are not of circular section, and hence may not be linked with simple analytic curves in the p-plane. What is clear, however, is that the locus of complex zeros of either of these wave functions may be determined, these zeros will appear in conjugate pairs since we are considering, for the moment, only real values of γ_2, and therefore the only contribution from the new integration path comes from the conjugate zeros of the wave function. The possibility of real plane wave contributions to these refracted fields does not arise, since with $M < |p| < 1$, this being the range of real values of p where we have seen the possibility of contributions, neither of the refracted wave functions is real.

This completes the discussion of the scattered field in the case of fully supersonic steady motion, except for the following remarks. When $h = 0$, the distinction between the singular reflected and refracted wave surfaces vanishes; they are all of circular section in the transverse plane and all residue contributions may be written down explicitly. Likewise if we restrict attention, for $h \neq 0$, to the free surface, the wave functions for the refracted field lose one of the radicals that makes calculations complicated, and again all contributions at the surface may be evaluated explicitly.

For subsonic values of the source velocity, a great deal of simplicity is lost. As the source velocity a is reduced from an infinite value, the point $p = M$ migrates; for $a > c_1$, M decreases in value from $m = c_2/c_1 < 1$ to zero, it becomes negative imaginary when $c_1 > a > c_2$, and it becomes real, approaching the value 1. From above, when $c_2 > a \to 0$. The parameter L merely changes from real when $a > c_1$ to positive imaginary when $a < c_1$, while the solution $p = R$ of equation (3) has the complex behaviour of γ_2 as long as the source velocity is greater than c_R. When $a < c_R$, it might be expected that like the parameter M, R is real, but with $R > 1$; however, with the branches chosen, the function $R(p)$ of equation (3) is not able to vanish in the correct range. This merely means that for sources which move steadily with a velocity below that of Rayleigh waves, the function $R(p)$ has no zeros, and thus there is no possibility for Rayleigh waves to be set up.

Apart from this point which is given special mention, there are no surprises in the form of singular surfaces. The singular dilatation surfaces vanish when $a < c_1$, the singular shear surfaces vanish when $a < c_2$, and the wedge-shaped Rayleigh singularity present on the surface for a surface source vanishes if $a < c_R$. The other parts of the calculation are complicated, both by the fact that the Cauchy principal value of the integrals on the locus of zeros of the wave function is no longer pure imaginary, and by the fact that shifting the integration path involves the calculation of loop integrals about the branch points $p = \pm M$. Given these integral contributions, one might do just as well, in general, by evaluating the original integrals on the formal integration path. If, however, there is special interest in fields close to points and surfaces of singularity, then the shifting of path is useful, because the complex zeros of the wave functions appear to provide as residues the most singular part of the field near the moving source, while the real double zeros of the wave functions will give information about field behaviour near the wave envelopes.

<u>Section 2:</u> Surface displacements.

Given the integrals (1.16) and (1.17) as general representations for the velocity potentials ϕ and $\underline{\psi}$, we may calculate both the velocity and the displacement field vectors without difficulty, by carrying out the appropriate differentiation and integration processes within the integral sign. Since the displacement on the free surface is the physical quantity of most interest, we shall restrict attention to the vicinity of the plane $y = 0$.

With the source buried, the potentials of interest are obtained by adding the integrals (1.16a) and (1.17). The velocity vector $\underline{v}$ is $\nabla\phi + \nabla \times \underline{\psi}$; this implies that the velocity field, linked in part with shear and in part with dilatation, is homogeneous and of degree -2 in space and time variables. The corresponding displacement vector $\underline{u}$ is therefore represented by homogeneous wave functions of degree -1.

For the buried supersonic source, the surface displacement is reduced to three terms. These are given by the equation

$$\underline{u} = -\mathrm{Rl}\int_{-\infty}^{\infty} \frac{2(1+L^2M^2)F\{2(p\underline{i}+LM\underline{k})(1-p^2)^{\frac{1}{2}}+\underline{j}(1-L^2M^2-2p^2)\}}{\gamma_2\, R(p)\,[\gamma_2\tau - px - h(M^2-p^2)^{\frac{1}{2}}]}\,dp \tag{6a}$$

$$-\mathrm{Rl}\int_{-\infty}^{\infty} \frac{4(pC-LMA)\{(p\underline{i}+LM\underline{k})[1-L^2M^2-2p^2-2(M^2-p^2)^{\frac{1}{2}}]-\underline{j}(1+L^2M^2)(M^2-p^2)^{\frac{1}{2}}\}}{\gamma_2\, R(p)\,[\gamma_2\tau - px - h(1-p^2)^{\frac{1}{2}}]}\,dp \tag{6b}$$

$$+\mathrm{Rl}\int_{-\infty}^{\infty} \frac{2\underline{i}\,[LMB - C(1-p^2)^{\frac{1}{2}}] + 2\underline{k}\,[A(1-p^2)^{\frac{1}{2}} - Bp]}{\gamma_2(1-p^2)^{\frac{1}{2}}\,[\gamma_2\tau - px - h(1-p^2)^{\frac{1}{2}}]}\,dp \tag{6c}$$

The integral (6a) is derived from the P waves produced by the source, while the other two integrals are derived from the S waves produced by the source. The integral (6b) is seen to contain the function $R(p)$ in the denominator, while the integral (6c) does not contain this factor. The significance of this property, to be brought out later in more detail, is that the integral 6c makes no Rayleigh wave contributions of any kind, although some effects of this kind are produced by (6b) .

When the source is taken on the surface, the displacement field has to be derived from the limiting forms, as $h \to 0$, of the potentials (1.16b) and (1.17) . At this point apparently the most concise way of defining the surface displacement is to take the limit as $h \to 0$ of the integrals (6) and then to add the correcting integrals

$$Rl \int_{-\infty}^{\infty} \frac{(p\underline{i} + LM\underline{k})(F - F^*) + \underline{j}(M^2 - p^2)^{\frac{1}{2}}(F^* + F)}{\gamma_2(\gamma_2 \ (\gamma_2\tau - px)(M^2 - p^2)^{\frac{1}{2}}} \, dp \tag{7a}$$

$$+ \ Rl \int_{-\infty}^{\infty} \frac{\underline{i}[(C^* + C)(1 - p^2)^{\frac{1}{2}} + LM(B^* - B)] + \underline{j}[LM(A - A^*) - p(C - C^*)] + \underline{k}[p(B - B^*) - (1 - p^2)^{\frac{1}{2}}(A + A^*)]}{\gamma_2 (\gamma_2\tau - px)(1 - p^2)^{\frac{1}{2}}} \, dp \tag{7b}$$

this contribution is, however, identically zero.

When the source velocity a is taken in the intermediate range $c_1 > a > c_2$, the velocity parameter M takes negative imaginary values as the transverse velocity of P waves becomes imaginary. While this is the only formal change to be made in the integrals (6) and (7), it will be found that there are line integral terms to complicate the solution as well as residue terms. When the source velocity is fully subsonic, besides the changes in the velocity parameters which follow the change of both the transverse velocities from real to imaginary values, the formal integration path is changed from the real to the imaginary axis, and an extra imaginary constant i is present as a factor in the numerator, just as in the integrals (1.10b) and (1.12b) .

In the fully supersonic case, with F a real function of p, the integral (6a) is the sum of conjugate residues at complex zeros of the wave function, at any rate in the case $h \neq 0$. The limiting form of this sum as $h \to 0$ must also include a residue contributions from one of the Rayleigh zeros $p = \pm R$, because in this case one of these points lies on the locus of zeros of the wave function.

Thus for the buried source, the surface displacement linked with the primary dilatation field is

$$\frac{-4\pi\gamma_1}{c_2^2(\gamma_1^2\tau^2 - x^2 - h^2)^{\frac{1}{2}}} \,\mathrm{Rl}\left\{\frac{(M^2 - p^2)^{\frac{1}{2}}[2(p\underline{i} + LM\underline{k})(1 - p^2)^{\frac{1}{2}} + \underline{j}(1 - L^2M^2 - 2p^2)]F}{F(p)}\right\}_{p = p_1} \qquad (8a)$$

for
$$p_1 = \frac{\gamma_2\tau x - iMh[\gamma_1^2\tau^2 - h^2 - x^2]^{\frac{1}{2}}}{x^2 + h^2}$$

and with $|x| < (\gamma_1^2\tau^2 - h^2)^{\frac{1}{2}}$. For the surface source we take the limit of the expression (8a) as $h \to 0$, and the additional Rayleigh residue contribution is

$$\frac{4\pi^2}{c_2^2} \,\mathrm{Rl}\left\{\frac{[2i(p\underline{i} + LM\underline{k})(p^2 - 1)^{\frac{1}{2}} + (1 - L^2M^2 - 2p^2)\underline{j}]F}{(dR/dp)\ \operatorname{sgn} x}\right\}_{p = R \operatorname{sgn} x} \delta[\tau - |x|/\gamma_R] \ , \qquad (8b)$$

(with the radicals in F taking positive imaginary values).

$$\frac{-8\pi}{\gamma_2[\gamma_2^2\tau^2-x^2-h^2]^{\frac{1}{2}}}\,\mathrm{Rl}\left\{\frac{(pC-LMA)(1-p^2)^{\frac{1}{2}}\{[1-L^2M^2-2p^2-2(M^2-p^2)^{\frac{1}{2}}](p\underline{i}+LM\underline{k})-\underline{j}(1+L^2M^2)(M^2-p^2)^{\frac{1}{2}}\}}{R(p)}\right\}_{p=p_2} \tag{9a}$$

with

$$p_2 = \frac{\gamma_2\tau x - ih(\gamma_2^2\tau^2 - x^2 - h^2)^{\frac{1}{2}}}{x^2 + h^2} \quad .$$

When $|x| > (\gamma_2^2\tau^2 - h^2)^{\frac{1}{2}}$, the real zeros of the wave function of (6b) give the head wave contribution

$$\frac{-8\pi\,\mathrm{sgn}\,x}{(x^2+h^2-\gamma_2^2\tau^2)^{\frac{1}{2}}}\,\mathrm{Im}\left\{\frac{(pC-LMA)(1-p^2)^{\frac{1}{2}}\{(p\underline{i}+LM\underline{k})[1-L^2M^2-2p^2-2(M^2-p^2)^{\frac{1}{2}}(1-p^2)^{\frac{1}{2}}]-\underline{j}(1+L^2M^2)(M^2-p^2)^{\frac{1}{2}}\}}{\gamma_2\,R(p)}\right\}_{p=p_H} . \tag{9b}$$

with $p_H = [\gamma_2\tau x - h\,\mathrm{sgn}\,x(h^2+x^2-\gamma_2^2\tau^2)^{\frac{1}{2}}]/(x^2+h^2)$ and with the radical $(M^2-p^2)^{\frac{1}{2}}$ taking the specific value, $i(p_H^2 - M^2)^{\frac{1}{2}}$, at this point. It is implicit that the head wave contribution only appears when $|p_H| > M$; this limits the expression for the surface displacement to the range $(\gamma_2^2\tau^2 - h^2)^{\frac{1}{2}} < |x| < \gamma_1\tau - h(1-M^2)^{\frac{1}{2}}/M$.

In addition to the contributions from 9a and 9b, taken in the limit as $h \to 0$, the field for the shear source the surface has as a contribution from the Rayleigh pole, the singular displacement

$$\frac{8\pi^2}{\gamma_2^2}\,\mathrm{Rl}\left\{\frac{(pC-LMA)\{(p\underline{i}+LM\underline{k})[1-L^2M^2-2p^2+2(p^2-M^2)^{\frac{1}{2}}(p^2-1)^{\frac{1}{2}}]-\underline{j}(1+L^2M^2)(p^2-M^2)^{\frac{1}{2}}\underline{i}\}}{(dR/dp)\,\mathrm{sgn}\,x}\right\}\delta[\tau-|x|/\gamma_R] . \tag{9c}$$

for $p = R \operatorname{sgn} x$, with positive imaginary values for the radicals in the bracket $pC - LMA$.

These representations are complete. Special notice may be taken of the singular contributions for the surface source field due to the singular point $p = R \operatorname{sgn} x$; in general we find both a localised singularity in the form of a delta function (as in (8b) and (9c), and a singularity (from (8a) and (9a)) which is locally antisymmetric about the point $|x| = \gamma_R \tau$. These contributions are not present in the case of the buried source because the integration path does not then pass through the points $p = \pm R$. However, when we evaluate the residues (8a) and (9a) at any point within the circles $|p - R \operatorname{sgn} x| = R - 1$, we may replace the function $R(p)$ by its Taylor expansion with the leading term of $0(p - R \operatorname{sgn} x)$. For a given x and h the minimum value of this factor occurs when $\gamma_2 \tau = R|x|$; this minimum value is small if $h << x$, and it is found that a surface disturbance whose amplitude is $0(x/h)$ will pass a fixed point with the velocity of Rayleigh waves, with its peak seen for $\gamma_R \tau = |x|$.

Without numerical work it is not possible to say whether this maximum is noticeable when h/x is not small. It is however implied in the formulae (8a) and (9a) that the quantities p_1 and p_2 are complex; the application of Taylor series to the residue (8a) is therefore restricted to the range $h < |x|(R^2 - 1)^{\frac{1}{2}}$. These limits have been mentioned elsewhere (Ewing et al., 1957), and they differ from the empirical result given by Pekeris (1957).

Part 3: The transient moving source in an elastic solid with a free plane surface.

Section 1: The simplification of double integral solutions.

For the steady supersonic source we have given a fairly detailed description of the process by which integrals of singular plane waves may be reduced to simple residue calculations. For the double integrals associated with a transient source, the evaluation is much more difficult; with two integration parameters there is no unique choice of complex paths, for each parameter, on which the wave function takes real values.

Fortunately, while we are dealing with propagation in an isotropic medium, we are able to determine a large amount of information in the manner described below. The simplest situation involves the reduction of the point source fields in the absence of a boundary. In the integrals (1.21b) or (1.23) we have specific integrals for the dilatation potential of a fixed transient source. In these integrals we are free to shift the 'slowness' parameters $(\frac{\alpha_1}{c_2}, \frac{\alpha}{c_2}, \frac{\alpha_3}{c_2})$ around the surface of a sphere of radius $m = c_1/c_2$; we may put $\alpha_1 = (m^2 - p^2)^{\frac{1}{2}} q$ and $\alpha_3 = (m^2 - p^2)^{\frac{1}{2}} (1 - q^2)^{\frac{1}{2}}$, so that $\alpha = p$, and we are to evaluate the integral

$$\mathrm{Rl} \iint_{-\infty}^{\infty} \frac{i\, F(p, q)}{(1 - q^2)^{\frac{1}{2}} \{c_2 t - (m^2 - p^2)^{\frac{1}{2}} [qx + (1 - q^2)^{\frac{1}{2}} z] - py \operatorname{sgn} y\}} \, dp\, dq \, , \qquad (1)$$

where, in the case of a source of unit strength,

$$F = 1/8\pi^3 \, , \qquad (2a)$$

and, in the case of a point force

$$F = -\{(m^2 - p^2)^{\frac{1}{2}}[qX + (1-q^2)^{\frac{1}{2}}Z] + pY \operatorname{sgn} y\}/8\pi^3 c_2 \quad . \tag{2b}$$

For $x = r\cos\theta$, $z = r\sin\theta$, we may choose a complex path on which

$$q = \cosh(w+i\theta) \operatorname{sgn} z, \quad (1-q^2)^{\frac{1}{2}} = -i \sinh(w+i\theta) \operatorname{sgn} z \quad ,$$

so that

$$qx + (1-q^2)^{\frac{1}{2}}z = r \operatorname{sgn} z \cosh w \quad .$$

On this path the integral (1) has the form

$$\phi = -\int_{-\infty}^{\infty} dw \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{F[p, \cosh(w+i\theta) \operatorname{sgn} z] + F[p, \cosh(w-i\theta) \operatorname{sgn} z]}{[c_2 t - (m^2 - p^2)^{\frac{1}{2}} r \cosh w \operatorname{sgn} z - py \operatorname{sgn} y} \, dp. \tag{3}$$

As long as the function $F(p, q)$ is a real function on the real axis of both variables when $|q| < 1$ and $|p| < m$, (as it is in the specific cases given in equation (2)), there is a real contribution to the integral (3) only from residues at conjugate poles of the integrand. The integration path for p is taken, just as in the simpler case of steady motion, along the locus of complex zeros of the demonimator for fixed values of x, y, z as t is varied. The principal value of the integral along this path is easily shown to be the difference of conjugates and hence imaginary, but the residue contributions from the zeros of the denominator give a real expression. Thus then the complex zero in the fourth quadrant of the p-plane is at the point $p = p_1$ where

$$p_1 = \frac{c_2 ty \operatorname{sgn} y - irm \cosh w\,(c_1^2 t^2 - y^2 - r^2 \cosh^2 w)^{\frac{1}{2}}}{y^2 + r^2 \cosh^2 w} , \tag{4}$$

this being a possibility only if, for $c_1 t > R = (r^2 + y^2)^{\frac{1}{2}} > 0$, w is restricted to the range for which $\cosh w < (c_1^2 t^2 - y^2)^{\frac{1}{2}}/r$. The evaluation of residues at p_1 and its conjugate point gives the result that

$$\phi = -2\pi \int_0^{w_1} Rt \frac{[(m^2 - p_1^2)^{\frac{1}{2}} \{F[p_1, \cosh(w + i\theta) \operatorname{sgn} z] + F[p_1, \cosh(w - i\theta) \operatorname{sgn} z]\}]}{m(c_1^2 t^2 - y^2 - r^2 \cosh^2 w)^{\frac{1}{2}}} dw \tag{5}$$

with $w_1 = \operatorname{arccosh}(c_1^2 t^2 - y^2)^{\frac{1}{2}}/r$. This is the only contribution for the fixed transient source. With F given as in (2), the finite integration can be carried out explicitly. Note that the only contribution in these integrals is confined to the interior of the wave surface $R = c_1 t$, which is a singular surface for the integral (1) because it is the envelope of the singular plane waves given in that expression. The same treatment leads to the potential for a source moving with subsonic velocity, the whole of the field being contained within the spherical wave surface $R = c_1 t$. Thus if we take the case of a transient source which moves along the y-axis with a velocity a we have to examine the integral

$$\phi = \mathrm{Ri} \iint_{-\infty}^{\infty} \frac{i\, c_2\, F(p_1 q)}{(1 - q^2)^{\frac{1}{2}} [c_2 - a \operatorname{sgn} y\, p] \{c_2 t - (m^2 - p^2)^{\frac{1}{2}} [qx + (1 - q^2)^{\frac{1}{2}} z] - py \operatorname{sgn} y\}} dp\, dq \ . \tag{6}$$

When $a < c_1$, the pole $p = c_2 \operatorname{sgn} y/a$ can be identified with a specific value of p_1 only when $r = 0$ and $y = at$, and this merely gives us a singular value at the source point for the complete field in the form

$$\cdot 2\pi \int_0^{w_1} \frac{c_1\, dw}{[c_1^2 t^2 - y^2 - r^2 \cosh^2 w]^{\frac{1}{2}}} \text{ Rl}\left\{\frac{(m^2-p_1^2)^{\frac{1}{2}}}{(c_2 - a \operatorname{sgn} y\, p_1)}\left[F[p_1, \cosh(w+i\theta) \operatorname{sgn} z] + F[p_1, \cosh(w-i\theta) \operatorname{sgn} z]\right]\right\}. \tag{7}$$

When $a > c_1$, the same pole lies between the branch points $p = \pm m$, and it coincides with a real zero $p = p_2$ of the denominator of the integrand, where

$$p_2 = \frac{c_1 ty \operatorname{sgn} y \pm mr \cosh w (y^2 + r^2 \cosh^2 w - c_1^2 t^2)^{\frac{1}{2}}}{y^2 + r^2 \cosh^2 w} .$$

The point about these real zeros is that neither is capable of generating a real residue contribution to the integral (3) or (5) unless the remaining factor of the integrand is non-real, or singular, at the point concerned. There is an important choice to be made here. For the integral (6) only the smaller value of p_2 can make a contribution in this manner; this is the value which lies to the left of the locus of complex zeros for a given y, r. Thus when, for $y > 0$ and $R > c_1 t$, we are able to find a p_2 with the value c_2/a and we may take the residue at this point in the negative sense to obtain the additional term

$$\phi = -\frac{2\pi\gamma_1}{a} \text{ Rl} \int_{-\infty}^{\infty} \frac{F(c/a,\, q)}{(1-q^2)^{\frac{1}{2}}\, [\gamma_1 (t - y/a) - qx - z(1-q^2)^{\frac{1}{2}}]}\, dq \quad . \tag{8}$$

This term is clearly a steadily moving supersonic field trailing behind the source point and contained within the singular conical front defined by the equation

$$r = \gamma_1(t-y/a) > 0, \; \gamma_1 = ac_1/(a^2-c_1^2)^{\frac{1}{2}} \;,$$

this cone being a tangential surface to the sphere $R = c_1 t$.

The reason for choosing the pole to lie to the left of the complex integration path is that this restricts the conical contribution to the region between the wave surface $R = c_1 t$ and the source point. If the other real zero of the wave function is involved, it produces the complementary conical field trailing behind the sphere $R = c_1 t$, but only for $y > 0$. This is not permissible because it involves discontinuity across the fixed surface $y = 0$, and this plane is not a characteristic surface of the system.

To make the same point more explicit, we should state that in the integral (1), the formal integration path for q is along the real axis except for deformations below $q = -1$ and above $q = 1$, and that for p is also along the real axis, but passing below the point $p = -m$ and above $p = m$, above the smaller value of p_2 and below the larger value of p_2 .

The importance of the choice of this path is made more evident if we change variables of integration, putting $P = (m^2-p^2)^{\frac{1}{2}}$ and $(m^2-P^2)^{\frac{1}{2}} = p$. Then with

$$\phi = \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{i\,P\,F[(m^2-P^2)^{\frac{1}{2}},\, q]\, c_2\, dp\, dq}{(1-q^2)^{\frac{1}{2}}(m^2-P^2)^{\frac{1}{2}}[c_2-a(m^2-P^2)^{\frac{1}{2}}\operatorname{sgn} y][c_2 t-P(qx+(1-q^2)^{\frac{1}{2}}z)-(m^2-P^2)^{\frac{1}{2}}y\operatorname{sgn} y]} \tag{9}$$

the supersonic contribution of this integral is found to be linked with the real zero of the wave function which lies to the right of the complex integration path and which coincides with the point $P = c_2/\gamma_1$ for $y > 0$. For the purpose of deciding the sense of integration at complex poles we may take the formal integration path for P along the real axis, below the point $P = -m$, above the point $P = m$, and below the smaller and above the larger of the real zeros

$$p = \frac{c_2 tr \cosh w \pm my(y^2 + r^2 \cosh^2 w - c_1^2 t^2)}{y^2 + r^2 \cosh^2 w}$$

The next point of difficulty in the evaluation of the scattered field is connected with the presence of the characteristic Rayleigh function

$$R = (1 - 2\alpha_1^2 - 2\alpha_3^2)^2 + 4\,\alpha\beta(\alpha_1^2 + \alpha_3^2)$$

which, because it has its own zeros, can contribute its own singularities to parts of the field. If we choose

$$\alpha_1 = pq, \quad \alpha_3 = p(1 - q^2)^{\frac{1}{2}}$$

with

$$\alpha = (m^2 - p^2)^{\frac{1}{2}}, \quad \beta = (1 - p^2)^{\frac{1}{2}} \tag{10}$$

then

$$R = R(p) = (1 - 2p^2)^2 + 4p^2(1 - p^2)^{\frac{1}{2}}(m^2 - p^2)^{\frac{1}{2}} \quad ,$$

and this function is easily recognised to have zeros at the points

$$p = \pm c_2/c_R \;,$$

c_R being the velocity of Rayleigh waves.

The characteristic function R only appears in the expressions 1.27 for the scattered field in the region $y < 0$. The complex integration path for these expressions will only pass through these Rayleigh zeros when for a surface source with $h = 0$ the point of observation is also on the surface with $y = 0$. Only in this case do we find a singular surface wave associated with the transient source. For the source which is initiated at a depth h, there will be found, when $h/r << 1$, a disturbance, which travels with velocity c_R , whose amplitude is $O(r/h)$. This point arises just as in the case of steady source motion. Even when the buried source moves towards the surface after it is set up, there is no singular surface wave until the source arrives at the surface, and then only because whatever may happen to the source there is bound to be a transient process at this moment which will set up its own spherical wave surfaces.

We have been discussing the various difficulties which arise in the evaluation of the dilatation potentials. The same difficulties also arise in the discussion of the shear potentials, but there is one more special situation, linked with the presence of head waves.

Head waves arise in the reflected shear field part of the integral (1.27b) . If we make the choice of α_1 and α_3 as given in the equations (10), it is clear that although the wave function contains only the branch points $p = \pm 1$, the remaining factors of the integral (specifically the Rayleigh function R) contain the branch points $p = \pm 1$ and $p = \pm m$. We have now the possibility that real zeros of the wave function can appear in the range $m < |p| < 1$, where the Rayleigh function is complex, and from these zeros we can find real contributions to the potential. The head wave contribution arises because the refracted P field travels faster along the surface than the associated shear field, and is incapable of satisfying the boundary conditions there by itself. The support for the head wave field is thus on the plane $y = 0$, and therefore it is real zero of the wave function which lie between the branch point $p = m \operatorname{sgn} z$ and the complex locus

$$p = \frac{c_2 tr \cosh w \operatorname{sgn} z \pm iy\,(c_2^2 t^2 - y^2 - r^2 \cosh^2 w)^{\frac{1}{2}}}{y^2 + r^2 \cosh^2 w} \quad ,$$

that is the zero

$$p_H = \frac{[c_2 tr \cosh w + y \ (y^2 + r^2 \cosh^2 w - c_2^2 t^2)^{\frac{1}{2}}]}{y^2 + r^2 \cosh w} \operatorname{sgn} z \quad ,$$

which gives a real residue contribution, with the range of w restricted to keep $|p_H| > m$ when $(y^2 + r^2)^{\frac{1}{2}} = R > c_2 t$.

There remain only computational difficulties in the evaluation of the total field due to a transient source. The primary source fields and the reflected fields can all be written down explicitly as finite integrals with respect to w , the ease with which these integrals can be found being linked with the simple nature of the associated singular wave fronts (they are either spherical or conical). The refracted wave fronts are not so simple to define without the use of parameters, but because the complex zeros of the wave functions arise in conjugate pairs, there is no doubt about the reduction of the p-integration to a residue calculation; these zeros however are found as solutions of a quartic algebraic equation.

Finally it will be repeated here that the structure of the scattered field is determined by the singular surfaces for the integrals (1.27), and these surfaces are determined by the condition that the denominator and its derivatives with respect to both integration parameters vanish together. There is no difficulty in this calculation.

Section 2: Calculations of surface displacements for a vertical force.

We may derive expressions for the surface displacement due to the general point source with a homogeneous potential of degree -1 . The potentials (1.26) and (1.27) are first differentiated to form the velocity and then integrated with respect to time to form integrals for the surface displacement. For the transient source which appears at a depth h, we find three distinct integrals; these are, first

$$\underline{u}_P = -\frac{2}{c_2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{iFV_1\{2\alpha[\alpha_1 \underline{i} + \alpha_3 \underline{k}] + (1-2\alpha_1^2 - 2\alpha_3^2)\underline{j}\}}{R(\alpha_1,\alpha_3)\,[c_2 t - \alpha_1 x - \alpha_3 z - \alpha h]}\, d\alpha_1\, d\alpha_3 \quad , \qquad (11a)$$

second

$$\underline{u}_{SR} = \frac{4}{c_2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{iV_2[c\alpha_1 - A\alpha_3]\{(\alpha_1\underline{i} + \alpha_3\underline{k})(1-2\alpha_1^2 - 2\alpha_3^2 - 2\alpha\beta) - \underline{j}\,\alpha\}}{R(\alpha_1,\alpha_3)\,[c_2 t - \alpha_1 x\, \alpha_3 z - \beta h]}\, d\alpha_1\, d\alpha_3 \quad , \qquad (11b)$$

and last

$$\underline{u}_S = -\frac{2}{c_2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{iV_2\{[B\alpha_3 - C\beta]\,\underline{i} + [A\beta - B\alpha_1]\underline{k}\}}{\beta[c_2 t - \alpha_1 x - \alpha_3 z - \beta h]}\, d\alpha_1\, d\alpha_3 \quad . \qquad (11c)$$

Of these integrals, the first is linked directly with the primary P field; the others, linked directly with the primary S field, differ in that the last is a simple horizontal displacement field which contains neither head wave nor surface wave contributions, while (11b) will be shown to contain both these contributions when appropriate. (The reason for the simplicity of (11c) is that the potentials A, B and C may contain neither the radical $\alpha = (m^2 - \alpha_1^2 - \alpha_3^2)^{\frac{1}{2}}$ nor the characteristic Rayleigh function $R(\alpha_1, \alpha_3)$.) These integrals, with $h = 0$, are correct expressions for the surface displacement when the transient source is set up on the surface.

The pure dilatation source, with F constant, $A = B = C = 0$, is the simplest case to evaluate, but since a head wave disturbance is not present for the buried source, certain aspects of the process of evaluation of the integrals (11) are not brought out. The simplest case we can take for a detailed examination is that of the transient vertical force, acting at a fixed point. For the vertical force Y we have the potentials

$$F = -(m^2 - p^2)^{\frac{1}{2}}\, Y/8\pi^3 c_2, \; B = 0 \;\text{ and }\; A\underline{i} + C\underline{k} = -Yp[(1-q^2)^{\frac{1}{2}}\underline{i} - q\underline{k}]/8\pi^3 c_2 \;, \tag{12}$$

with $V_1 = V_2 = 1$, these being the forms when $\alpha_1 = pq$, $\alpha_3 = p(1-q^2)^{\frac{1}{2}}$.

The integrals to be evaluated are

$$\underline{u}_P = \frac{Y}{4\pi^3}\, \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{ip(m^2-p^2)^{\frac{1}{2}}\{\underline{j}(1-2p^2) + 2p(1-p^2)^{\frac{1}{2}}[q\underline{i} + (1-q^2)^{\frac{1}{2}}\underline{k}]\}}{(1-q^2)^{\frac{1}{2}} R(p)\{c_2 t - p[qx + (1-q^2)^{\frac{1}{2}} z] - (m^2-p^2)^{\frac{1}{2}} h\}}\, dq\, dq \;, \tag{13a}$$

$$\underline{u}_{SR} = -\frac{Y}{2\pi^3 c_2^2} \iint_{-\infty}^{\infty} \frac{ip^3\{p[q\underline{i} + (1-q^2)^{\frac{1}{2}}\underline{k}][1 - 2p^2 - 2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}] - \underline{j}(m^2-p^2)^{\frac{1}{2}}\}}{(1-q^2)^{\frac{1}{2}} R(p)\{c_2 t - p[qx + (1-q^2)^{\frac{1}{2}} z] - (1-p^2)^{\frac{1}{2}} h\}}\, dp\, dq \;, \tag{13b}$$

and

$$\underline{u}_S = -\frac{Y}{4\pi^3 c_2^2} \,\mathrm{Rl} \iint_{-\infty}^{\infty} \frac{ip^2 [q\underline{i} + (1-q^2)^{\frac{1}{2}} \underline{z}] - (1-p^2)^{\frac{1}{2}} \underline{h}\}}{(1-q^2)^{\frac{1}{2}} \{c_2 t - p[qx + (1-q^2)^{\frac{1}{2}} z] - (1-p^2)^{\frac{1}{2}} h\}} \, dp\, dq \;. \tag{13c}$$

Now take the integration path for q into the complex plane; with

$$x = r\cos\theta, \quad z = r\sin\theta, \quad q = \cosh(w + i\theta) \text{ and } (1-q^2)^{\frac{1}{2}} = -i\sinh(w+i\theta) \;;$$

(with a restriction to $z > 0$ in order to simplify the writing down without losing any generality) we find that on this path the contribution from $w < 0$ is the complex conjugate of the contribution for $w > 0$. Hence the q-integration may be reduced, in part, to an integration for positive w , with the contributions

$$\underline{u}_P^{(1)} = -\frac{Y}{2\pi^3 c_2^2} \int_0^{\infty} dw \,\mathrm{Rl} \int_{-\infty}^{\infty} \frac{p(m^2-p^2)^{\frac{1}{2}}[(1-2p^2)\underline{j} + 2p(1-p^2)^{\frac{1}{2}} \cosh w \hat{\underline{r}}]}{R(p)\,[c_2 t - pr\cosh w - (m^2-p^2)^{\frac{1}{2}} h]} \, dp \;, \tag{14a}$$

where $\hat{\underline{r}}$

$$\underline{u}_{SR}^{(1)} = \frac{Y}{\pi^3 c_2^2} \int_0^{\infty} dw \,\mathrm{Rl} \int_{-\infty}^{\infty} \frac{p^3\{p\cosh w[1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}]\hat{\underline{r}} - (m^2-p^2)^{\frac{1}{2}}\underline{j}\}}{R(p)\,[c_2 t - pr\cosh w - (1-p^2)^{\frac{1}{2}} h]} \, dp \;, \tag{14b}$$

and

$$\underline{u}_S = \frac{Y}{2\pi^3 c_2^2} \int_0^{\infty} dw \,\mathrm{Rl} \int_{-\infty}^{\infty} \frac{p^2 \cosh w\, \hat{\underline{r}}}{c_2 t - pr\cosh w - (1-p^2)^{\frac{1}{2}} h} \, dp \;, \tag{14c}$$

following simply from the integrals (13) .

The integration with respect to p may be carried out in the usual manner. Thus in (14a), the only real contribution to the integral comes from residues at the conjugate complex zeros of the wave function, and these complex zeros are only present under the restriction that $\cosh w < (c_1^2 t^2 - h^2)^{\frac{1}{2}}/r$. Thus

$$\underline{u}_P^{(1)} = \frac{\underline{Y}}{\pi^2 c_2^2} \int^{\operatorname{arccosh}[c_1^2 t^2 - h^2]^{\frac{1}{2}}/r} \frac{dw}{(c_1^2 t^2 - h^2 - r^2 \cosh^2 w)^{\frac{1}{2}}} \; \mathrm{Rl}\,\{G_1(p_1)\} \quad ,$$

where

$$p_1 = \frac{c_2\, tr \cosh w - imh\,(c_1^2 t^2 - h^2 - r^2 \cosh^2 w)^{\frac{1}{2}}}{h^2 + r^2 \cosh^2 w} \quad ,$$

where both the radicals $(1-p^2)^{\frac{1}{2}}$ and $(m^2 - p^2)^{\frac{1}{2}}$ have positive real parts when $p = p_1$, and where

$$G_1(p) = -\frac{\underline{p}(m^2 - p^2)\,[\underline{j}(1-2p^2) + 2\underline{\hat{r}}\;\underline{p}(1-p^2)^{\frac{1}{2}} \cosh w]\,\underline{Y}}{m\,R(p)} \quad .$$

This integral is only present when $0 \leq r \leq (c_1^2 t^2 - h^2)^{\frac{1}{2}}$. To reduce it to a simpler form, we may make the real transformation

$$c_1^2 t^2 - h^2 - r^2 \cosh^2 w = (c_1^2 t^2 - h^2 - r^2) \sin^2 \lambda \quad ,$$

and we have the result that

$$\underline{u}_P^{(1)} = \frac{1}{\pi^2 c_2^2 r} \int_0^{\pi/2} \mathrm{Rl}\,\{G_1(p_1)\} \tag{15a}$$

within the singular P circle $r = (c_1^2 t^2 - h^2)^{\frac{1}{2}}$.

With

$$p^2 = \frac{c_2 tr \cosh w - ih\,(c_2^2 t^2 - h^2 - r^2 \cosh^2 w)^{\frac{1}{2}}}{h^2 + r^2 \cosh^2 w} \quad ,$$

with positive real parts specified for the radicals and under the transformation

$$c_2^2 t^2 - h^2 - r^2 \cosh^2 w = (c_2^2 t^2 - h^2 - r^2) \sin^2 \mu \quad ,$$

the shear field contributions from complex zeros of the wave function are

$$\underline{u}_{SR}^{(1)} = \frac{1}{\pi^2 c_2^2 r} \int_0^{\pi/2} \frac{d\mu}{\cosh w} \mathrm{Rl}\,\{G_2(p_2)\} \tag{15b}$$

with

$$G_2(p) = \frac{Y\{2p^4(1-p^2)^{\frac{1}{2}} \cosh w\,[1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}]\hat{\underline{r}} - 2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}} p^3 \underline{j}\}}{R(p)}$$

and

$$\underline{u}_S = \frac{1}{\pi^2 c_2^2 r} \int_0^{\pi/2} \frac{d\mu}{\cosh w} \, \mathrm{Rl}\,\{G_3(p_2)\} \quad , \tag{15c}$$

with

$$G_3(p) = Y_p^2(1-p^2)^{\frac{1}{2}} \cosh w \, \hat{\underline{r}} \quad ,$$

inside the singular S circle $r = (c_2^2 t^2 - h^2)^{\frac{1}{2}}$.

For the buried source with $h \neq 0$, the integrals (15a) and (15c) are the only contribution to the surface displacement to be derived from the integrals (14a) and (14c) . The integral (14b), on the other hand, contains both the contribution (15b) linked with complex poles of the integrand and a head wave contribution as well. There is a real residue contribution from the real pole $p = p_H$ where

$$p_H = \frac{c_2 tr \cosh w - h(r^2 \cosh^2 w + h^2 - c_2^2 t^2)^{\frac{1}{2}}}{h^2 + r^2 \cosh^2 w} \quad ;$$

this contribution appears only when $p_H > m$. This condition imposes the restriction that $r \cosh w < c_1 t - h(1 - m^2)^{\frac{1}{2}}/m$. With the change of integration variable given by

$$r^2 \cosh^2 w + h^2 - c_2^2 t^2 = (r^2 + h^2 - c_2^2 t^2) \cosh^2 \Lambda$$

we find that for $(c_2^2t^2 - h^2)^{\frac{1}{2}} < r < c_1 t - h(1-m^2)^{\frac{1}{2}}/m$,

$$\underline{u}_{SR}^{(1)} = \frac{1}{\pi^2 c_2^2 r} \int_0^{\Lambda_1} \frac{d\Lambda}{\cosh w} \operatorname{Im}\{G_2(p_H)\} \quad , \tag{15d}$$

with

$$\Lambda_1 = \operatorname{arccosh}\left\{\frac{c_2 t(1-m^2)^{\frac{1}{2}} - h}{m(r^2+h^2-c_2^2t^2)^{\frac{1}{2}}}\right\} \quad .$$

This is the head wave contribution for the buried source.

Notice that for points on the compressional wave front, $p_1 = m(1-h^2/c_1^2t^2)^{\frac{1}{2}}$ is constant and the integral (15a) takes the simple value

$$\underline{u}_P^{(1)} = G_1(p_1)/2\pi\, c_2^2 r \quad ,$$

with $w = 0$. On the shear front $p_2 = (1-h^2/c_2t^2)^{\frac{1}{2}}$ is constant, and the integrals (15b) and (15c) similarly that the simple values

$$\underline{u}_{SR}^{(1)} = G_2(p_2)/2\pi\, c_2^2 r$$

and

$$\underline{u}_S = G_3(p_2)/2\pi\, c_2^2 r \quad ,$$

with $w = 0$. These simple expressions are the initial displacements on the surface linked with the arrival of P and S waves; they agree with the results of Pekeris (1957) for the horizontal components of displacement, but for the vertical component they do not. (His initial P displacement lacks the factor $(1-h^2/c_1^2t^2)^{\frac{1}{2}}$, and his initial S displacement lacks the factor $h(1-h^2/c_2^2t^2)^{\frac{1}{2}}/c_2t$.)

The integral for the head wave is also simplified at the front of the S wave. The integrand (15d) reduces to a constant, but the integral becomes logarithmically infinite together with the upper limit, Λ_1, of integration. Pekeris also notes this logarithmic singularity; we differ in that he places it just inside the S wave, instead of just outside, and in that his formula for this logarithmic singularity in the vertical component lacks the factor $(1-h^2/c_2t^2)^{\frac{1}{2}}$.

For the surface source, the integrals (15) have to be evaluated with $h = 0$. This is not a uniform limit of the case for the buried source because the integration path involves real values of p to the right of $p = m$, thus passing through the Rayleigh pole $p = R$. Thus with the total surface displacement field (14) written in the more concise form

$$\underline{u} = -\frac{Y}{2\pi^3 c_2^2}\,\mathrm{Rl}\int_0^\infty dw \int_{-\infty}^{\infty} \frac{\underline{p}(m^2-p^2)^{\frac{1}{2}}\underline{j} - [1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}]\,p^2\,\underline{\hat{r}}\cosh w}{R(p)\,[c_2t - pr\cosh w]}\,dp \tag{16}$$

and with the p-integration path shifted to the horizontal loop to the right of $p = m$ we find a number of distinct contributions. These are

$$\underline{u} = -\frac{2Y}{\pi^2 c_2^2} \int_m^{c_2 t/r} \frac{(1-2p^2)(p^2-m^2)^{\frac{1}{2}} p[(1-2p^2)\underline{i} + (1-p^2)^{\frac{1}{2}} \hat{\underline{r}} c_2 t/r]}{[(1-2p^2)^4 + 16p^4(1-p^2)(p^2-m^2)](c_2^2 t^2 - p^2 r^2)^{\frac{1}{2}}} dp \qquad (17a)$$

when $m < c_2 t/r < 1$, and

$$\underline{u} = -\frac{2Y}{\pi^2 c_2^2} \int_m^1 \frac{p(1-2p^2)^2 (p^2-m^2)^{\frac{1}{2}} (1-p^2)^{\frac{1}{2}} \hat{\underline{r}} c_2 t/r}{[(1-2p^2)^4 + 16p^4(1-p^2)(p^2-m^2)](c_2^2 t^2 - p^2 r^2)^{\frac{1}{2}}} dp \qquad (17b)$$

$$-\frac{2Y}{\pi^2 c_2^2} \int_1^{c_2 t/r} \frac{p(p^2-m^2)^{\frac{1}{2}} \underline{i}}{[(1-2p^2)^2 - 4p^2(p^2-1)^{\frac{1}{2}}(p^2-m^2)^{\frac{1}{2}}](c_2^2 t^2 - p^2 r^2)^{\frac{1}{2}}} dp \qquad (17c)$$

$$-\frac{2Y}{\pi c_2^2} \frac{R \hat{\underline{r}} c_2 t/r}{D(R)(c_2^2 t^2 - R^2 r^2)^{\frac{1}{2}}} \qquad (17d)$$

where $D(R) = [dR(p)/dp]_{p=R} = -8R[6R^4(1-m^2) + 2R^2(2m^2-3) + 1](1-2R^2)^{-2}$,

when $c_2 t/r > 1$ for (17b), (17c), and $c_2 t/r > R$ for (17d) .

The expressions (17a) and (17b) come from the contribution to (16) of the range $m < p < 1$, (17c) comes from the range $p > 1$, and (17d) comes from the single point $p = R$. No other real terms are present. The integrals (17a) and (17c) contain a singularity at $p = R$ if $c_2 t/r > R$.

The integrals are easily identified with those given by Pekeris (1955a); we differ in the value of the coefficient of the final Rayleigh wave term. It is noted that the integrals (17) may be simplified; the vertical component may be evaluated explicitly and the horizontal component reduces to forms involving incomplete elliptic integrals. This work has been done by Pekeris (1955a). [What is most interesting is that the surface displacement field may be found in its simplest form by an alternative integration process (for (13)) involving first the calculation of residue contributions from the zeros of $R(p)$ due to <u>all</u> the branches of the radicals $(1-p^2)^{\frac{1}{2}}$ and $(m^2-p^2)^{\frac{1}{2}}$, and then the evaluation of the q-integral in the usual manner. For the vertical component the q-integration is simply a matter of calculating residues, and this is a simple way of noting the existence of a lacunà, already seen by Pekeris. The horizontal component, however, involves quadratures. The alternative method of integration which is centred on the existence of multiple sheets for the Rayleigh slowness surface will be discussed elsewhere.]

Notice finally that the appearence of Rayleigh singularities for the surface source is accompanied by the simultaneous reduction in the order of singularity linked with the arrival of P and S waves. Where for the buried source we expect a step function singularity for the P wave, or a step function coupled with a conjugate logaritmic singularity for the S wave, the surface source field has the same singularities in its first derivatives, and is therefore continuous in its displacement components.

<u>Section 3</u>: Surface displacements for fixed sources with an unsymmetrical displacement field; the horizontal force and the couple with arbitrary orientation.

The value of the present method is that the evaluation of the surface displacement field from the integrals (11) is not made any more difficult if there is no axis of symmetry. Having just considered the case of the vertical force, with its obvious axis of symmetry, we shall now examine the case where a horizontal force $X\underline{i}$ is applied at a point below the surface of an elastic solid. With the potentials

$$F = -Xpq/8\pi^3 c_2, \quad A = 0, \quad B = Xp(1-q^2)^{\frac{1}{2}}/8\pi^3 c_2, \quad C = -X(1-p^2)^{\frac{1}{2}}/8\pi^3 c_2 \tag{18}$$

the integrals 11 take the form

$$\underline{u}_P = \frac{X}{4\pi^3 c_2^2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{ip^2 q\{2p(1-p^2)^{\frac{1}{2}}[q\underline{i} + (1-q^2)^{\frac{1}{2}}\underline{k}] + (1-2p^2)\underline{j}\}}{R(p)(1-q^2)^{\frac{1}{2}}\{c_2 t - p[qx(1-q^2)^{\frac{1}{2}} z] - h(m^2-p^2)^{\frac{1}{2}}\}} \, dp\, dq \quad , \tag{19a}$$

$$\underline{u}_{SR} = -\frac{X}{2\pi^3 c_2^2} \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{ip^2 q(1-p^2)^{\frac{1}{2}}\{p[q\underline{i} + (1-q^2)^{\frac{1}{2}}\underline{k}][1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}] - \underline{j}(m^2-p^2)^{\frac{1}{2}}\}}{R(p)\,(1-q^2)^{\frac{1}{2}}\{c_2 t - p[qx + (1-q^2)^{\frac{1}{2}} z] - h(1-p^2)^{\frac{1}{2}}\}} \, dq\, dq, \tag{19b}$$

and

$$\underline{u}_S = \frac{X}{4\pi^3 c_2^2} \mathrm{Rl} \iint \frac{ip^3 q[q\underline{i} + (1-q^2)^{\frac{1}{2}}\underline{k}]}{(1-p^2)^{\frac{1}{2}}(1-q^2)^{\frac{1}{2}}\{c_2 t - p[qx + (1-q^2)^{\frac{1}{2}} z] - h(1-p^2)^{\frac{1}{2}}\}} \, dp\, dq \quad . \tag{19c}$$

It follows, with the abbreviations $\cosh w = c$, $\sinh w = s$, $\underline{\hat{r}} = \underline{i}\cos\theta + \underline{k}\sin\theta$ and $\underline{\hat{\theta}} = \underline{k}\cos\theta - \underline{i}\sin\theta$, that

$$G_1(p)\ m\ R(p) = -Xp^2(m^2-p^2)^{\frac{1}{2}}\left\{(1-2p^2)c\cos\theta\,\underline{i} + 2p(1-p^2)^{\frac{1}{2}}[c^2\underline{\hat{r}}\cos\hat{\theta} + s^2\theta\sin\theta]\right\}, \tag{20a}$$

$$G_2(p)R(p) = 2Xp^2(1-p^2)\left\{-(m^2-p^2)^{\frac{1}{2}}c\cos\theta\,\underline{i} + p[1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}]\cdot\right.$$
$$\left.\cdot[c^2\underline{\hat{r}}\cos\theta + s^2\,\underline{\hat{\theta}}\sin\theta]\right\}, \tag{20b}$$

and

$$G_3(p) = Xp\,[\underline{\hat{r}}\cos\theta\,(1-p^2c^2) - \underline{\hat{\theta}}\sin\theta\,(1+p^2s^2)], \tag{20c}$$

these being the three expressions needed in the integrals (15) to define the surface displacement for the buried source.

For the surface source we have the additional term associated with the residue at the point $p = R$. The complete contribution to the surface displacement of this residue is restricted to the region $r < c_R t$, with the form

$$-\frac{Xt[1-2R^2+2(R^2-1)^{\frac{1}{2}}(R^2-m^2)^{\frac{1}{2}}]\underline{i}}{\pi c_2^2 r D(R)\ (c_R^2t^2-r^2)^{\frac{1}{2}}}$$

The field due to the sudden application of a couple is of interest to seismologists; this is the point source model of symmetrical shear in the focus of tectonic earthquakes, with symmetrical displacement in relation to a plane of rupture, and with movement of the sides of this plane in opposite directions (Keilis-Borok 1960). The couple is therefore to be taken as the limit of a double force with moment, with the force acting in the plane of rupture.

We specify that the couple is of moment M, that it acts in the plane $z = 0$, and that the component forces in this plane act at an angle ϵ to the plane $y = 0$. For the couple of step function time dependence, the velocity potential and the displacement field are both functions of degree -2 in space and time variables. To keep within the structure of the calculations of this paper, we shall be considering the surface displacements given in the equations (11) and these must subsequently be differentiated with respect to t. It might be suggested that the result for a couple is easily found by taking the results for a force and differentiating in the appropriate direction; this process is however available only when we have explicit results for the field both on and off the free surface.

Thus we use the potentials

$$F = \frac{M}{8c_2^2\pi^3}[pq\cos\epsilon + (m^2-p^2)^{\frac{1}{2}}\sin\epsilon][pq\sin\epsilon - (m^2-p^2)^{\frac{1}{2}}\cos\epsilon] \;,$$

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{M}{8c_2^2\pi^3}[pq\sin\epsilon - (1-p^2)^{\frac{1}{2}}\cos\epsilon] \begin{Bmatrix} p(1-q^2)^{\frac{1}{2}}\sin\epsilon \\ -p(1-q^2)^{\frac{1}{2}}\sin\epsilon \\ (1-p^2)^{\frac{1}{2}}\cos\epsilon - pq\sin\epsilon \end{Bmatrix} \;, \qquad (21)$$

in the integrals (11), and we derive the quantities

$$\frac{G_1(p)2mc_2R(p)}{p(m^2-p^2)^{\frac{1}{2}}M} = \left\{\begin{array}{l} [2p(1-p^2)^{\frac{1}{2}}c\hat{\underline{r}}+(1-2p^2)\underline{1}]\left\{\begin{array}{l}\sin 2\epsilon[p^2c^2\cos 2\theta - p^2s^2\sin^2\theta + p^2 - m^2] \\ -2(m^2-p^2)^{\frac{1}{2}}\cos\theta\cos 2\epsilon\end{array}\right\} \\ + \\ 2p^2(1-p^2)^{\frac{1}{2}}s^2\hat{\underline{\theta}}\sin\theta[2(m^2-p^2)^{\frac{1}{2}}\cos 2\epsilon - pc\cos\theta\sin 2\epsilon] \end{array}\right\},$$

$$\frac{G_2(p)c_2R(p)}{2Mp^2(1-p^2)^{\frac{1}{2}}} = \begin{array}{l} \{pc\hat{\underline{r}}[1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}] - \underline{1}(m^2-p^2)^{\frac{1}{2}}\}\cdot \\ \cdot\left\{\begin{array}{c} p(1-p^2)^{\frac{1}{2}}s^2\sin^2\theta\sin\epsilon\cos\epsilon \\ + \\ [(1-p^2)^{\frac{1}{2}}\cos\epsilon - pc\cos\theta\sin\epsilon][c(1-p^2)^{\frac{1}{2}}\cos\theta\cos\epsilon - p\sin\epsilon] \end{array}\right\}, \\ -ps^2\sin\theta\hat{\underline{\theta}}[1-2p^2-2(1-p^2)^{\frac{1}{2}}(m^2-p^2)^{\frac{1}{2}}]\cdot \\ \cdot\left\{\begin{array}{l} p\sin\epsilon[c(1-p^2)^{\frac{1}{2}}\cos\theta\cos\epsilon - p\sin\epsilon]+ \\ + s(1-p^2)^{\frac{1}{2}}\cos\epsilon[(1-p^2)^{\frac{1}{2}}\cos\epsilon - pc\cos\theta\sin\epsilon] \end{array}\right\}, \end{array}$$

and

$$\frac{G_3(p)c_2}{Mp} = \begin{array}{l} [pc\cos\theta\sin\epsilon - (1-p^2)^{\frac{1}{2}}\cos\epsilon]\left\{\begin{array}{c} \hat{\underline{r}}[p(1-p^2)^{\frac{1}{2}}c\sin\epsilon - (1-p^2c^2)\cos\theta\cos\epsilon] \\ + \\ \theta(1+p^2c^2)\cos\epsilon\sin\theta \end{array}\right\} - \\ -p^2s^2\sin\theta\sin\epsilon\{\hat{\underline{r}}cp\cos\epsilon\sin\theta - \hat{\underline{\theta}}[pc\cos\epsilon\cos\theta + (1-p^2)^{\frac{1}{2}}\sin\epsilon] \end{array}. \quad (22)$$

The functions $G_1(p)$, $G_2(p)$ and $G_3(p)$ are used in the time derivative of the integrals (15) in the calculation of the complete field for the fixed buried couple. Note that where for the buried force the initial P and S field is a simple discontinuity, i.e., a step function of displacement, the couple will be associated with initial delta function displacements.

Section 4: Summary, with a description of the singular parts of the surface displacement for the general point source.

In the earlier parts of this paper, we have discussed the details of calculation for the surface displacements produced by a point source. The usual problem in which this information may be useful is the one of recognising the type of source which produces a given displacement field. It is in this context that knowledge of initial displacements is of use; we shall therefore conclude this paper by giving results for the singular parts of the surface displacement field for the general point source.

The velocity potential for the general point source of order $(n+1)$ is a homogeneous function of degree $-(n+1)$ in space and time variables; it is given, for $y > 0$, by the integrals

$$\phi = \left(\frac{\partial}{\partial t}\right)^n \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{i F V_1}{\alpha[c_2 t - \alpha_1 x - \alpha_3 z - \alpha y]} \, d\alpha_1 \, d\alpha_3 \quad ,$$

$$\underline{\psi} = \left(\frac{\partial}{\partial t}\right)^n \mathrm{Rl} \iint_{-\infty}^{\infty} \frac{i V_2 [A\underline{i} + B\underline{j} + C\underline{k}]}{B[c_2 t - \alpha_1 x - \alpha_3 z - \beta y} \, d\alpha_1 \, d\alpha_3 \quad ,$$

with $\alpha = (\dot{m}^2 - \alpha_1^2 - \alpha_3^2)^{\frac{1}{2}}$ and $\beta = (1 - \alpha_1^2 - \alpha_3^2)^{\frac{1}{2}}$.

The functions

$$V_1 = \frac{c_2}{(c_2 - \alpha_1 v_1 - \alpha_3 v_3 - \alpha v_2)}$$

and

$$V_2 = \frac{c_2}{(c_2 - \alpha_1 v_1 - \alpha_3 v_3 - \beta v_2)}$$

introduce the effect of steady motion for $t > 0$, the source being forced to move with the point $x = v_1 t$, $y = v_2 t$, $z = v_3 t$. The prototype of these potentials is related to the force $\rho(X\underline{i} + Y\underline{j} + Z\underline{k})$; when this acts at the origin we have a first order source for which

$$-8\pi^3 c_2 F = X\alpha_1 + Y\alpha + Z\alpha_3, \quad 8\pi^3 Ac_2 = Z\beta - Y\alpha_3, \quad 8\pi^3 Bc_2 = X\alpha_3 - Z\alpha_1 ,$$

and

$$8\pi^3 Cc_2 = Y\alpha_1 - X\beta .$$

Other point source fields may be obtained by differentiation or integration of the point force field. The general form given arises because for the functional dependence given, we always have the identities

$$\frac{\partial}{\partial x} = -\frac{\alpha_1}{c_2}\frac{\partial}{\partial t} , \quad \frac{\partial}{\partial z} = \frac{-\alpha_3}{c_2}\frac{\partial}{\partial t} ;$$

for a compressional field which moves in the direction of positive y

$$\frac{\partial}{\partial y} = -\frac{\alpha}{c_2}\frac{\partial}{\partial t} ,$$

and for a shear field

$$\frac{\partial}{\partial y} = -\frac{\beta}{c_2}\frac{\partial}{\partial t} .$$

For the initially buried source, with a velocity restricted to subsonic values, the surface field contains three circular fronts on which the displacement field or its derivatives are singular, these being associated for $r = (c_1^2 t^2 - h)^{\frac{1}{2}}$ with the arrival of the primary P waves, for $r = c_1 t - h(1-m^2)^{\frac{1}{2}}/m$ with the arrival of the refracted P waves and hence of the head waves of shear, and for $r = (c_2^2 t - h^2)^{\frac{1}{2}}$ with the arrival of the primary S waves.

For the first order source the first arrival is a step-function P wave, given from the integral 11a with an amplitude

$$\frac{4\pi^2 p(m^2-p^2)^{\frac{1}{2}} F(\alpha_1, \alpha_3) V_1(\alpha_1, \alpha_3) [2p(1-p^2)^{\frac{1}{2}} \underline{r} + (1-2p^2)\underline{1}]}{mc_2 R(p)}$$

where for $z > 0$, $\theta = \arctan z/x$, this is evaluated with $\alpha_1 = p \cos\theta$, $\alpha_3 = p \sin\theta$, $p = m(1-h^2/c_1 t^2)^{\frac{1}{2}}$ and with positive values for the radicals $(1-p^2)^{\frac{1}{2}}$ and $(m^2-p^2)^{\frac{1}{2}}$.

For the shear field there are the distinct situations linked with the ranges $h < c_2 t < h/(1-m^2)^{\frac{1}{2}}$ or $c_2 t > h/(1-m^2)^{\frac{1}{2}}$. For the former range, the refracted P field has not separated itself from the incident S field, there is thus no head wave of shear, and the arrival of shear is marked by a step function of amplitude

$$-\frac{4\pi^2(1-p^2)^{\frac{1}{2}} V_2(\alpha_1, \alpha_3)[C\cos\theta - A\sin\theta][p(1-2p^2)\hat{\underline{r}} - p^2(m^2-p^2)^{\frac{1}{2}}\underline{1}]}{c_2 R(p)}$$

$$-\frac{4\pi^2 V_2(\alpha_1, \alpha_3) B}{c_2} \hat{\underline{\theta}} \qquad ,$$

where V_2, A, B and C are evaluated at the points $\alpha_1 = p\cos\theta$, $\alpha_3 = p\sin\theta$ with $p = (1-h^2/c_2^2t^2)^{\frac{1}{2}}$. When head waves are present, there is no discontinuity in the displacement field associated with their arrival, this for the first order source. The same step function is present at the arrival of the primary S waves, but it is clearly masked by the conjugate logarithmic singularity associated with the head waves. This contribution has the form

$$\frac{16\pi\Lambda_1[V_2(C\cos\theta - A\sin\theta)]\{\underline{1}(1-2p^2)^2 + p(1-2p^2)(1-p^2)^{\frac{1}{2}}\hat{\underline{r}}\}(p^2-m^2)^{\frac{1}{2}}}{c_2\,[(1-2p^2)^4 + 16p^4(1-p^2)(p^2-m^2)]} ,$$

this just outside the circle $r = (c_2^2t^2 - h^2)^{\frac{1}{2}}$, with V, A and C again calculated at the points $\alpha_1 = p\cos\theta$, $\alpha_3 = p\sin\theta$, when $p = (1-h^2/c_2^2t^2)^{\frac{1}{2}} > m$ and with the logarithmically large quantity Λ_1 defined by the equation

$$\Lambda_1 = \operatorname{arccosh} \frac{c_2t(1-m^2)^{\frac{1}{2}} - h}{m(r^2+h^2-c_2^2t^2)^{\frac{1}{2}}} .$$

These are the singular arrivals for the first order source. Each differentiation increases the order of the source and makes the singular contributions more singular. For the source of order $(n+1)$, the leading singularity linked with the arrival of the head wave is the $(n-1)$-th derivative of a step function, i.e., the function $\delta^{(n-2)}$, with an amplitude proportional to the vector

$$\underline{1}(1-2m^2) + m\,\hat{r} .$$

The arrivals associated with the principal wave fronts are more singular. The primary arrivals are dominated by the nth derivative of a step function i.e., the function $\delta^{(n-1)}$, while the logarithmic head wave singularity leads to the conjugate function of degree $-n$, a continuous function which is $0(d^{-n})$ as the distance d from the S wave approaches zero in the head wave region. The relative magnitudes of the various components of displacement are the same as for the first order source.

If the source velocity is supersonic, we expect conical singular surfaces to trail behind the source point and to be tangential to the spherical wave surfaces linked to the setting up of the source. Only if the source point is outside the sphere $[r^2 + (y+h)^2]^{\frac{1}{2}} = c_1 t$ and above the tangent surface $r(c_1^2 t^2 - h^2)^{\frac{1}{2}} = c_1^2 t^2 - yh$ will the P cone cut the plane $y = 0$ to show a new singular wave on the surface; this will be a segment of an ellipse if the source point is below the lowest point $y = -h - c_1 t$ of the corresponding spherical P wave (i.e., if $v_2 < -c_1 t$) otherwise it is a segment of a hyperbola. Likewise only if the source point is outside the spherical S wave and above the tangent surface $r(c_2^2 t^2 - h^2)^{\frac{1}{2}} = c_2^2 t^2 - h^2$ will the S cone produce its effect on the free surface, and again if $v_2 < -c_2$ there is a singular wave of elliptical form on the surface, and this becomes hyberbolic if $v_2 > -c_2$.

These singularities in the form of conic sections are of the type found in the description of steady conical fields. For the P wave due to a source of order 1, we find specific displacements on the singular front in the form

$$\frac{-4\pi^2 v_r\, G(\alpha_1, \alpha_3)\, \psi_2}{a^2[c_2^2(t-\xi/a)^2 - (m^2 - c_2^2/a^2)(\zeta^2+\eta^2)]^{\frac{1}{2}}}$$

where $v_r = (v_1^2 + v_3^2)^{\frac{1}{2}}$, $a = (v_r^2 + v_2^2)^{\frac{1}{2}}$, and

$$G(\alpha_1, \alpha_3) = [2\alpha(\alpha_1 \underline{i} + \alpha_3 \underline{k}) + (1 - 2\alpha_1^2 - 2\alpha_3^2)\underline{j}]\ \frac{F(\alpha_1, \alpha_3)}{R(\alpha_1, \alpha_3)}$$

with

$$\alpha_1 = \frac{v_1 v_r \psi_1 - v_1 v_2 \psi_2 - a v_3 \psi_3}{a v_r}, \quad \alpha_3 = \frac{v_3 v_r \psi_1 - v_3 v_2 \psi_2 + a v_1 \psi_3}{a v_r}$$

and with

$$a\psi_1 = c_2, \ \psi_3 = \frac{(a^2 - c_1^2)^{\frac{1}{2}}\zeta}{c_1(ta-\xi)}, \quad \psi_2 = \frac{(a^2-c_1^2)^{\frac{1}{2}}\eta}{c_1(ta-\xi)}$$

for

$$a\xi = v_1 x + v_2 h + v_3 z \quad ,$$

$$a v_r \eta = -v_1 v_2 x + v_r^2 h - v_3 v_2 z \quad ,$$

and

$$a\zeta = -v_3 x + v_1 z \quad .$$

The singular displacement for the conical S wave is of the same type. We take the same values for α_1 and α_3, but we are to evaluate the expression

$$\frac{4\pi^2 v_r H(\alpha_1, \alpha_3)}{a^2[c_2^2(t-\xi/a)^2 - (1-c^2/a^2)(\zeta^2+\eta^2)]^{\frac{1}{2}}}$$

when $\psi_1 = c_2/a$, $\psi_3 = \dfrac{(a^2-c_2^2)^{\frac{1}{2}}}{c_2(at-\xi)}\zeta$, and $\psi_2 = \dfrac{(a^2-c_2^2)^{\frac{1}{2}}}{c(at-\xi)}\eta$

with

$$R(\alpha_1, \alpha_3) = \frac{2(C\alpha_1 - A\alpha_3)\{(\alpha_1 \underline{i} + \alpha_3 \underline{k})[1-2\alpha_1^2-2\alpha_3^2-2\alpha\beta] - \underline{i}\,\alpha\}}{R(\alpha_1, \alpha_3)} -$$

$$- \frac{(B\alpha_3 - C\beta)\,\underline{i} + (A\beta - B\alpha_3)\,\underline{k}}{\beta}$$

For a range of values which makes the radical $(m^2-\alpha_1^2-\alpha_3^2)^{\frac{1}{2}}$ imaginary, there is a head wave contribution; this fixes the head wave singularity in the form

$$\frac{4\pi^2 v_r I_m[H(\alpha_1, \alpha_3)]}{a^2[(1-c_2^2/a^2)(\zeta^2+\eta^2) - c_2^2(t-\xi/a)^2]^{\frac{1}{2}}} \quad .$$

The arrival of the head wave, linked with the arrival of refracted P waves, is not marked by discontinuities of displacement.

For the higher order source, differentiation with respect to t is necessary; the source of order $n+1$, moving steadily with supersonic velocity, has conical wave fronts on which the singularity is $0(d^{-n-\frac{1}{2}})$.

As a final contribution we shall discuss the source which appears on the surface. In addition to the limiting forms of expressions which are found for the buried source, the effect of the singularity due to the Rayleigh function is to give extra contributions in the form

$$\underline{u}_P^2 = \frac{8\pi^2\{(1-2R^2)\,\mathrm{Rl}(FV_1)\,\underline{i} - 2R(R^2-m^2)^{\frac{1}{2}}\,\mathrm{Im}(FV_1)\,\underline{\hat{r}}\}}{c_2\, d(R)\ (c_R^2t^2-r^2)^{\frac{1}{2}}}$$

and

$$\underline{u}_{SR}^{(2)} = \frac{16\pi^2\{R\,[1-2R^2+2(R^2-1)^{\frac{1}{2}}(R^2-m^2)^{\frac{1}{2}}]\,\mathrm{Rl}(\psi)\,\underline{\hat{r}} + (R^2-m^2)^{\frac{1}{2}}\,\mathrm{Im}(\psi)\,\underline{i}}{c_2\, D(R)\ (c_R^2t^2-r^2)^{\frac{1}{2}}}$$

within the circle $r = c_R t$. Here,

$$\psi = V_2(C\cos\theta - A\sin\theta) \ ,$$

is evaluated at $\alpha_1 = R\cos\theta$, $\alpha_3 = R\sin\theta$, with the radicals α, β taking positive imaginary values, and with $D(R)$ defined as for the equations (17) .

If the source moves on the surface with a velocity greater than c_R, a wedge-shaped Rayleigh wave is set up. If it moves with a supersonic velocity there are conical S and P waves set up; all the details of these steadily moving fields may be picked out of Part 2.

The surface source of order $n+1$ sets up transient P and S waves which are $O(d^{-n+1})$ near the wave front, while the conical P and S waves associated with steady supersonic motion are $0(d^{-n+\frac{1}{2}})$. Thus for the surface source the P and S arrivals are of the same singularity as the head wave arrival for the buried source.

The Rayleigh wave produced by the setting up of the source is $0(d^{-n-\frac{1}{2}})$ near the wave front, this being comparable with the arrival of steady conical P and S waves for a buried source. The strongest singularity is that of the steady wedge-shaped Rayleigh wave for the moving source, this being $0(d^{-n-1})$.

All other parts of the surface displacement field are infinitely differentiable.

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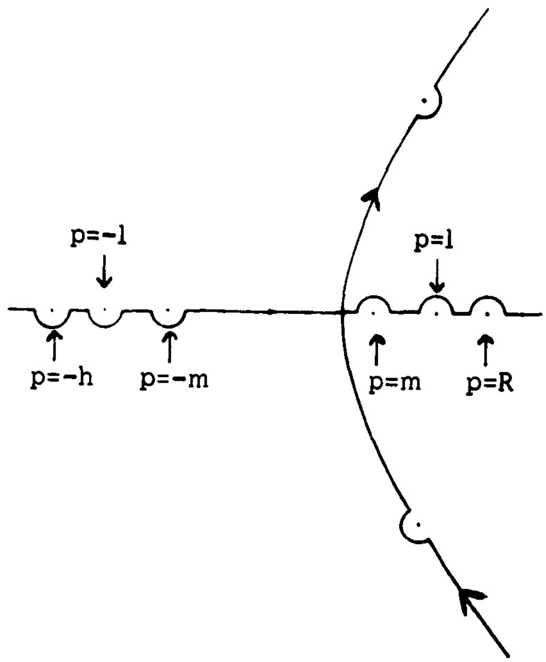


Figure 1a: The formal integration path passing below singularities at $p = -R$, -1, $-m$, and above those at $p = m$, 1 and R, together with the actual hyperbolic path of integration, deformed to pass round conjugate complex zeros of the wave function. Residues at these conjugate points determine the field inside the characteristic wave envelope.

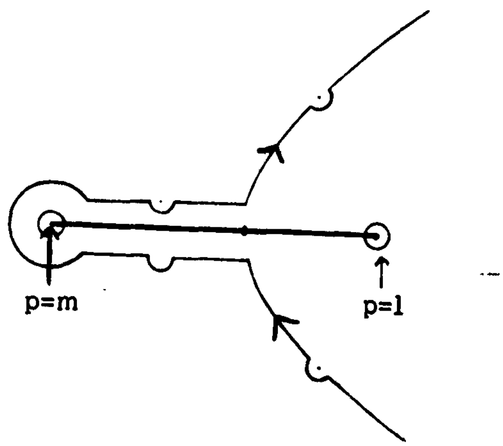


Figure 1b: When the locus of complex zeros passes between the points $p = m$, and $p = 1$, the actual integration path must be composed of a hyperbola with a horizontal loop round the branch point $p = m$. The complex zeros of the wave function again determine the field inside the characteristic wave envelope, while real zeros determine the field in a head wave region.

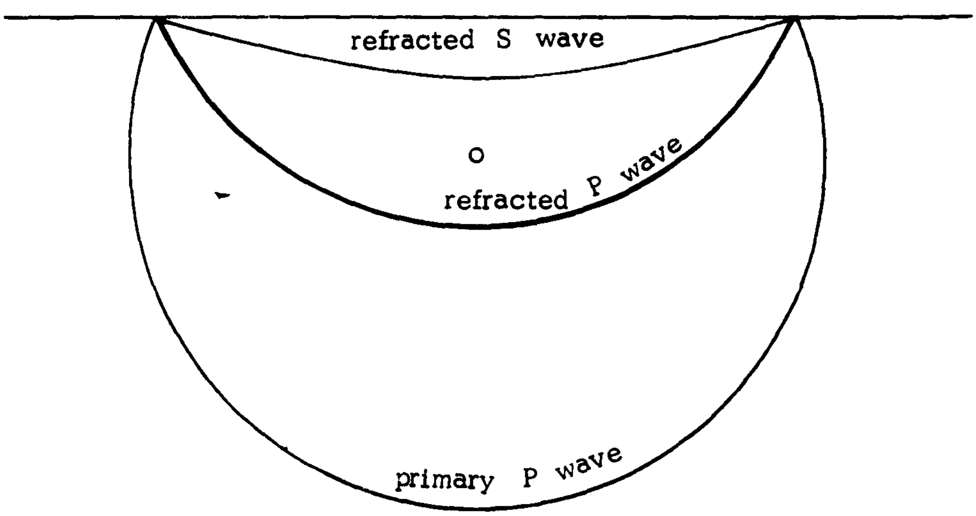
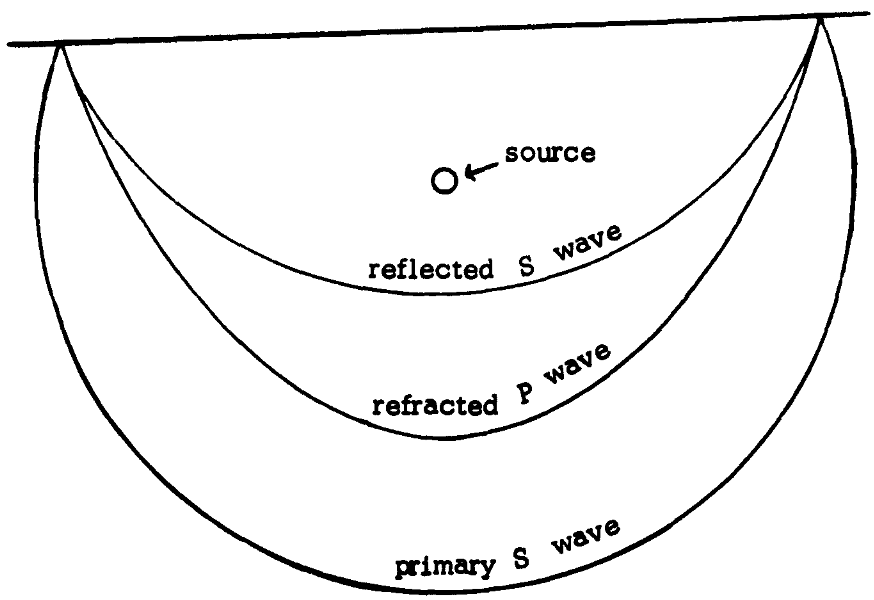


Figure 2: The structure of the field due to a primary source of dilatation.



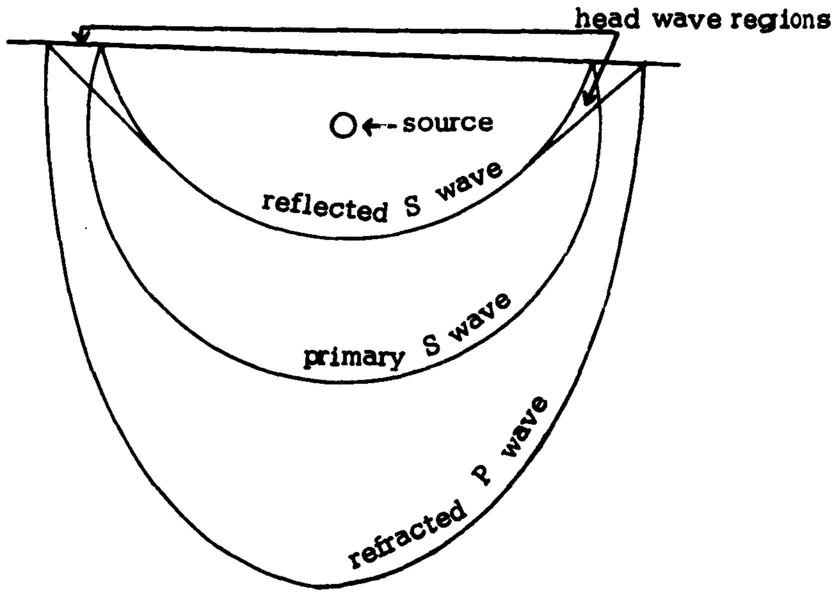


Figure 3: The structure of the field due to a primary source of shear. Two stages are shown. Before the wave front of the primary field begins to travel at the critical angle, the wave fronts both of reflected S and refracted P waves travel together along the free surface. At a certain instant the refracted P wave meets the free surface at right angles, and it may only continue to travel at the velocity of dilatation waves by breaking away from the incident and reflected S waves. Head waves of shear now appear in order to satisfy boundary conditions at the surface. The two stages shown indicate the initial development, and the subsequent breakaway of the refracted field.